1. (14 points) Short questions:

a) (1 point) If I am talking about something that can be everything, then I am talking about:

   i) signals
   ii) systems
   iii) signals and systems
   iv) something
   v) nothing

   Solution: The correct answer is (v). See notes from Lecture 2.

b) (2 points) Consider the system:

$$y(t) = 2x(t^2).$$

   Is the system linear?

   Solution: Yes. Suppose $x_1(t), x_2(t)$ are inputs to this system with outputs $y_1(t) = 2x_1(t^2)$ and $y_2(t) = 2x_2(t^2)$, respectively. Then for scalars $a, b \in \mathbb{R}$, let $\hat{x}(t) = ax_1(t) + bx_2(t)$. The associated output $\hat{y}(t)$ is thus:

$$\hat{y}(t) = 2\hat{x}(t^2) = 2(ax_1 + bx_2)(t^2) = 2ax_1(t^2) + 2bx_2(t^2) = ay_1(t) + by_2(t)$$

   Therefore, the system is linear.
   Remark: Some students did not specify the I/O relationship

c) (3 points) Consider the Foucault pendulum system discussed in lecture. The input $x(t)$ to the system is the applied force and the output of the system $y(t)$ is the angle the string makes with the vertical. They are related by the differential equation:

$$mr\frac{d^2y}{dt^2} + mg\sin y(t) = x(t).$$

   Suppose the applied force $x(t)$ is a sinusoid at frequency 100 Hz before time $t = 2$s and becomes a sinusoid at frequency 200 Hz after time $t = 2$s. Is the system time-invariant? Explain.

   Solution: Whether a system is time-invariant or not does not depend on a specific input being applied. A system is time-invariant if for any input-output pair $(x(t), y(t))$, the signals $(x(t-\tau), y(t-\tau))$ also form an input output pair for any $\tau \in \mathbb{R}$. This system was shown to be time-invariant in class as can be inferred from solutions of the differential equation.
   Remark: Some students got confused about signal and system

d) (4 points) A continuous-time speech signal has no frequency components beyond 20 kHz. It is sampled at the rate of $30,000$ samples per second. What is the highest non-zero frequency component (in Hz) of the continuous-time signal captured in the discrete-time samples? What about if the sampling rate is $50,000$ samples per second?

Solution: The highest non-zero frequency component represented would be half the sampling frequency or the highest non-zero frequency component present in the CT signal, whichever is smaller. So, the answers to above two questions are 15 kHz and 20 kHz respectively.

Remark: Some students did not solve for the period

e) (4 points) A periodic input signal of period p is fed into a LTI system. Is the output necessarily periodic? If not, give a counter-example. If so, give an argument to explain why and compute the period of the output signal.

Solution: Yes, the output will be periodic. The signal $x(n)$ will have a Fourier decomposition of the form $x(n) = \sum_{k=-K}^{K} C_k e^{i\cdot\frac{2\pi}{p}kn}$, where $K = \lceil \frac{p-1}{2} \rceil$. Now, an LTI system with input $f(n) = e^{i\omega n}$ has output $H(\omega)e^{i\omega n}$. So, by superposition the output of the LTI system with input $x(n)$ would simply be $\sum_{k=-K}^{K} H(\frac{2\pi}{p}k) C_k e^{i\cdot\frac{2\pi}{p}kn}$, which is easily seen to be periodic with period that divides p. The period may be p or some divisor of p. (For instance, if the LTI system averages the last p values, then the output becomes a constant and the period becomes 1.)

Alternative solution: Just using time invariance is enough. No need for linearity. Let $y(n)$ be the output of the system with input $x(n)$. By time invariance, the output of the system with input $x(n-k)$ would be $y(n-k)$ for any integer k. Let us choose $k = p$. Since $x(n)$ is periodic with period p, we have that $x(n-p) = x(n)$ for all n. Thus $x(n-p)$ and $x(n)$ are exactly the same input signal. So, $y(n-p)$ and $y(n)$ are the same output signal, so that $y(n)$ is periodic with period p or a divisor of p.

2. (8 points) You have measured the frequency response of H of a (real) LTI system at angular frequency 100 rad/s, i.e. you know $H(100)$. Based on this information alone, give the output to the input signal $x(t)$ in each of the parts below, or or explain why there is not enough information to compute it.

a) (2 points) $x(t) = \sin(100t + \frac{\pi}{4})$;

b) (2 points) $x(t) = \sin^2(100t)$;

c) (2 points) $x(t) = \sin(100t) + \cos(100t)$;

d) (2 points) $x(t) = e^{-i100t}$.

Solution: Since the system is real, $H(-100) = H(100)^*$. So knowing $H(100)$, we can determine $H(-100)$ as well.

a) $x(t) = \sin(100t + \frac{\pi}{4}) = \frac{e^{i\cdot 100t}e^{i\pi/4} - e^{-i\cdot 100t}e^{-i\pi/4}}{2i}$. Since the system is LTI, we will have $y(t) = \frac{H(100)e^{i\cdot 100t}e^{i\pi/4} - H(-100)e^{-i\cdot 100t}e^{-i\pi/4}}{2i}$ = imaginary part of the complex number $H(100)e^{i\cdot 100t + i\pi/4}$, or $|H(100)|\sin(100t + \frac{\pi}{4} + \angle H(100))$.

b) $x(t) = \sin^2(100t) = \frac{1-\cos(200t)}{2}$. Since we do not know $H(0)$ and $H(200)$, it is impossible to determine the output of the system based on $H(100)$ alone.

c) $x(t) = \sin(100t) + \cos(100t) = \frac{e^{i\cdot 100t} - e^{-i\cdot 100t}}{2i} + \frac{e^{i\cdot 100t} + e^{-i\cdot 100t}}{2}$. By superposition, the output of the system would be $y(t) = \frac{H(100)e^{i\cdot 100t} - H(-100)e^{-i\cdot 100t}}{2i} + \frac{H(100)e^{i\cdot 100t} + H(-100)e^{-i\cdot 100t}}{2}$.

d) $x(t) = e^{-i100t}$. This is a pure complex exponential. So, the output of the LTI system would be $y(t) = H(-100)e^{-i100t}$.

Remark:

Some students used $y(t) = x(t)H(100)$ for **everything, which is not correct in general.**

Some students did not know $H^*(100) = H(-100)$.

Some students missed $\angle H(100)$ in $|H(100)| \cdot \sin(100t + \angle H(100))$

Some sudents did not know the frequency of $\sin^2(100t)$

3. (7 points) Consider the following signal defined on the reals:

$$x(t) = \begin{cases} 1 & \text{if } t \in [n, n+0.5) \text{ for some even integer } n \text{ or } t \in [n, n+1) \text{ for some odd integer } n \\ -1 & \text{if } t \in [n+0.5, n+1) \text{ for some even integer } n \end{cases}$$

a) (2 points) Plot the signal $x(t)$ as a function of t, and label the key features.

b) (2 points) Is the signal periodic? If so, calculate the period. If not, explain.

c) (3 points) The signal is passed through a system with input-output relationship given by :

$$y(t) = 2x(\frac{t}{2} - 1).$$

Plot the output signal $y(t)$ for the given input, and label the key features.

Solution:

a) See Fig. 1.

Remark: Some students treated positive and negative time differently. Some students drew 2 plots , one for even n and one for odd n.

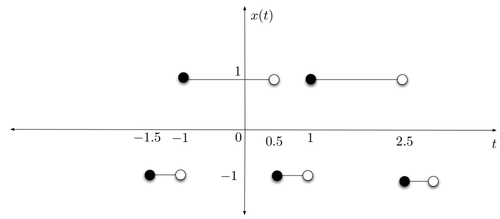

Figure 1: $x(t)$

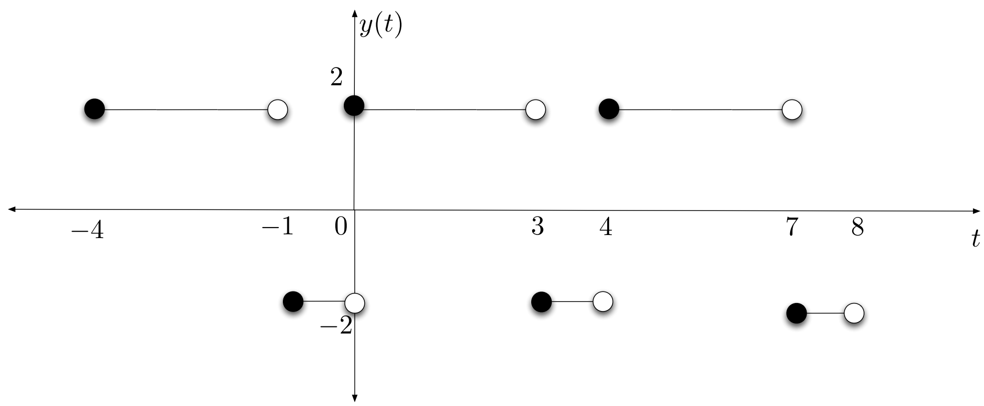

Figure 2: $y(t)$

b) Yes, the system is periodic with period 2.

c) See Fig. 2.

Remark: Some students scale the period by $\frac{1}{2}$ instead of 2. Some students got the signal delay wrong.

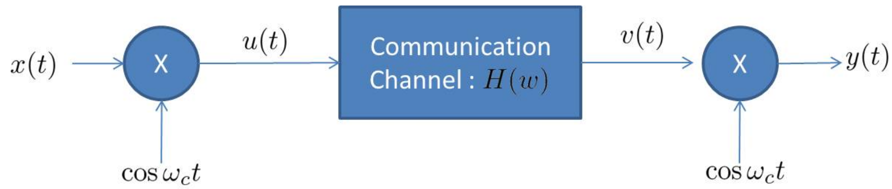

Figure 3: Diagram for the communication system

4. (6 points) Consider a discrete-time periodic signal $x(n)$ of period p. You can assume p is even.

   a) (2 points) Give the Fourier series expansion of $x(n)$ in terms of cosines.

   b) (4 points) From the expansion in part (a), derive the Fourier series expansion of $x(n)$ in terms of complex exponentials. Give the coefficients in terms of the coefficients of the expansion in part (a).

   Solution:

   a) Let $K = \frac{p}{2}$. Then, $x(n) = A_0 + \sum_{k=1}^{K} A_k \cos(\frac{2\pi}{p}kn + \phi_k)$.

   b) We know that $x(n)$ may also be expressed as $x(n) = \sum_{k=-K}^{K} C_k e^{i\frac{2\pi}{p}kn}$. These two expressions would be equal if we have $C_0 = A_0$ and for $1 \le k \le K$,

   $$C_k e^{i\frac{2\pi}{p}kn} + C_{-k} e^{i\frac{-2\pi}{p}kn} = A_k \cos(\frac{2\pi}{p}kn + \phi_k).$$

   This yields $C_k = C_{-k}^*$ and $\Re(C_k) = \Re(C_{-k}) = \frac{A_k \cos\phi_k}{2}$ and $\Im(C_k) = -\Im(C_{-k}) = \frac{A_k \sin\phi_k}{2}$.

   Remark: Some students did not realize $e^{-i\phi_k} \neq e^{i\phi_k}$.

5. (15 points) A communication system is depicted in Fig. 3. The communication channel is modeled by a LTI system with a frequency response given by:

   $$H(\omega) = \begin{cases} A & \text{if } \omega_c - \frac{W}{2} \le |\omega| \le \omega_c + \frac{W}{2} \\ 0 & \text{else} \end{cases}$$

   The frequency ω_c is called the *carrier frequency* and the parameter W is called the *bandwidth*. You can assume that $\omega_c \gg W$. For example, in cellular communication, the carrier frequency may be 1 GHz and the bandwidth may be 10 MHz.

   a) (2 points) Plot the frequency response of the channel, labeling the key features.

   b) (5 points) Suppose the input signal is a sinusoid $x(t) = \cos\omega t$ with $\omega \in (0, \frac{W}{2})$. Give explicit expressions for the signals $u(t), v(t)$ and$y(t)$.

c) (3 points) Consider the overall system with $x(t)$ as the input and $y(t)$ as the output. Is the system LTI? Why or why not?

d) (5 points) Suppose you are allowed to further process $y(t)$ by passing it through a LTI system to yield another output $\tilde{y}(t)$. Design the frequency response of this LTI system such that $\tilde{y}(t) = x(t)$.

Solution:

a) See Fig. 4.

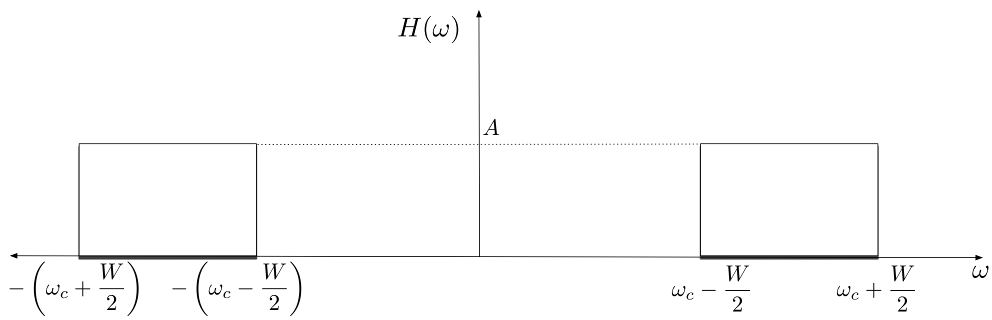

Figure 4: Frequency Response Magnitude. Phase is zero throughout.

Remark: Some students only plotted $|H(w)|$ for $w > 0$.

b) $u(t) = \cos(\omega t)\cos(\omega_c t) = \frac{1}{2}\left[\cos((\omega_c - \omega)t) + \cos((\omega_c + \omega)t)\right]$
$v(t) = \frac{A}{2}\left[\cos((\omega_c - \omega)t) + \cos((\omega_c + \omega)t)\right]$
$y(t) = \frac{A}{4}\left[2\cos(\omega t) + \cos((2\omega_c - \omega)t) + \cos((2\omega_c + \omega)t)\right]$
Note that the above expressions assume $\omega \in (0, \frac{W}{2})$.
Remark: Some students just wrote down $v(t) = H(w)u(t)$. Some students did not know why we should multiply $\cos \omega_c t$

c) The system is not LTI as the output signal has frequency components $2\omega_c - \omega$ and $2\omega_c + \omega$ which are completely absent in the input signal.
Remark: Most students think the system is LTI but it is **NOT.**

d) Consider a system with frequency response given by

$$\tilde{H}(\omega) = \begin{cases} \frac{2}{A} & \text{if } |\omega| < \frac{W}{2} \\ 0 & \text{else} \end{cases}$$

When $y(t)$ is fed as input to this system, the output will be $\tilde{y}(t) = x(t)$ for the chosen input.
Remark: Many students thought that $G(w) = \frac{1}{\cos \omega_c t}$ but it is **WRONG . We are looking for a LTI system,** and to be expressed in terms of its frequency response.