Solutions To Midterm # 1

1. $H_0 = \frac{L^2}{2I}$

a) Our eigenstates must be eigenstates of L^2, so they have eigenvalues

$$L^2 |\ell m\rangle = \ell(\ell+1)\hbar^2 |\ell m\rangle$$

$$\therefore \quad E_{\ell m} = \frac{(\ell)(\ell+1)\hbar^2}{2I} \equiv E_\ell$$

for a given ℓ, there are $(2\ell+1)$ possible values of m. So the degeneracy is $(2\ell+1)$ for each energy E_ℓ

b) $H' = \lambda \vec{L} \cdot \vec{S}$

we can write $\vec{J} = \vec{L} + \vec{S}$, so

$$J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$$

$$\therefore \quad L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$$

using $E_n^{(1)} = \langle \psi_n | H' | \psi_n \rangle$ we can find the 1^{st} order corrections

<u>for $\ell = 0$</u> $\qquad J$ must $= 1/2$

so $\quad \langle \ell = 0\ m = 0 | H' | \ell = 0\ m = 0 \rangle =$

$$\frac{1}{2}\left[(1/2)(3/2)\hbar^2 - 0 - (\tfrac{1}{2})(3/2)\hbar^2\right) = 0$$

there is <u>no</u> change to energy of ground state

degeneracy $= \underbrace{(2\ell+1)}_{\text{spatial}} \underset{\substack{\uparrow \\ \text{spin}}}{(2)} = (1)(2) = 2$

<u>for $\ell = 1$</u> $\qquad J$ can $= 3/2\,,\ 1/2$

for $J = 3/2$:

$$\langle J = 3/2\ m_J | H' | J = 3/2\ m_J \rangle =$$

$$\frac{1}{2}\left[(3/2)(5/2)\hbar^2 - (1)(2)\hbar^2 - (\tfrac{1}{2})(3/2)\hbar^2\right]$$

$$= \frac{1}{2}\left[\frac{15}{4} - \frac{8}{4} - \frac{3}{4}\right]\hbar^2 = \frac{1}{2}\hbar^2$$

$$\therefore \quad E = E^{(0)} + E^{(1)} = \frac{(1)(2)\hbar^2}{2I} + \frac{\hbar^2}{2}\lambda$$

For $J = 3/2 \qquad E = \dfrac{\hbar^2}{I} + \dfrac{\hbar^2}{2}\lambda$

degeneracy is : $(2J+1) = 4$

For $j = 1/2$

$\langle j = 1/2 \ m_j | H' | j = 1/2 \ m_j \rangle =$

$\frac{1}{2} \left[(1/2)(3/2)\hbar^2 - 1(2)\hbar^2 - 1/2(3/2)\hbar^2 \right]$

$= -\hbar^2$

$$\therefore \quad E = E^{(0)} + E^{(1)} = \frac{(1)(2)\hbar^2}{2I} - \hbar^2 \lambda$$

$$= \frac{\hbar^2}{I} - \hbar^2 \lambda$$

degeneracy is $(2j+1) = 2$

note: we started with $(2\ell+1) = 3$ spatial states $\times$ 2 spin states $= 6$ total states.

We ended with 4 $j = 3/2$ and 2 $j = 1/2$ $= 6$ total states. This is a good cross-check that we calculated the degeneracy properly.

2 a) False

The relativistic correction to kinetic energy is $-p^4/8m^3c^2$. Since p^4 always has a positive expectation value, $\Delta E^{(1)} = \langle \frac{-p^4}{8m^3c^2} \rangle$ is always less than 0.

b) False

Hyperfine splitting is due to the interaction of the 2 magnetic moments : $H \propto \vec{\mu}_1 \cdot \vec{\mu}_2$ where

$$\vec{\mu} = g \frac{e\hbar}{2m} \vec{S}$$

So hydrogen hyperfine spitting goes like $\frac{1}{M_p m_e}$ while positronium goes like $\frac{1}{m_e m_e}$. Since $M_p \gg m_e$, hyperfine splitting is much smaller for hydrogen.

c) False

$$H = \frac{p^2}{2m} + V(r) + \frac{e\hbar}{2m} \vec{L} \cdot \vec{B}$$

If B is in x direction

$$H = \frac{p^2}{2m} + V(r) + \frac{e\hbar}{2m} L_x B_x$$

For L_z to be conserved, $[H, L_z]$ would have to be ϕ. But

$$[L_x, L_z] = \phi$$

So L_z is not conserved. (L_x is conserved

True

d) Our wave function is a superposition of $|l=1\ m=1\rangle$ and $|l=1\ m=-1\rangle$ Since the state have a fixed value of l, it is an eigenstate of L^2 with eigenvalue $l(l+1)\hbar^2$

3a) $\hat{S}_z$ is diagonal and has the eigenvalu[es] of S_z as its elements

$$\therefore S_z = \begin{pmatrix} 2 & & & & \\ & 1 & & 0 & \\ & & 0 & & \\ & 0 & & -1 & \\ & & & & -2 \end{pmatrix}$$

b) $\langle S_z \rangle = \langle \psi | S_z | \psi \rangle$

$= \left(\sqrt{3/5}\, \langle 2\,2| + \sqrt{2/5}\, \langle 2\,1|\right) \hat{S}_z \left(\sqrt{3/5}\, |2\,2\rangle + \sqrt{\tfrac{2}{5}}\, |2\,1\rangle\right)$

$= \frac{3}{5}(2\hbar) + \frac{2}{5}(\hbar) = \frac{8}{5}\hbar$

c) Write $S_x = \dfrac{S_+ + S_-}{2}$

$S_\pm |s\, m_s\rangle = \sqrt{s(s+1) - m(m\pm1)}\;\hbar\, |s\, m_s \pm 1\rangle$

So S_+ only has non-~~ze~~zero elements one above the diagonal and S_- only has non-zero elements one below the diagonal

$$S_+ |1\ {-2}\rangle = \sqrt{2(3) - (-2)(-1)}\ \hbar\ |1\ {-1}\rangle = \sqrt{4}\ |1\ {-1}\rangle$$
$$= 2\ |1\ {-1}\rangle$$

$$S_+ |1\ {-1}\rangle = \sqrt{2(3) - (-1)(0)}\ \hbar\ |1\ 0\rangle = \sqrt{6}\ |1\ 0\rangle$$

$$S_+ |1\ 0\rangle = \sqrt{2(3) - (0)(1)}\ \hbar\ |1\ 1\rangle = \sqrt{6}\ |1\ 1\rangle$$

$$S_+ |1\ 1\rangle = \sqrt{2(3) - (1)(2)}\ \hbar\ |1\ 2\rangle = 2\ |1\ 2\rangle$$

$$S_+ |1\ 2\rangle = 0$$

$$\therefore \quad S_+ = \hbar \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

similarly

$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\frac{S_+ + S_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$