

**Final Examination**  
**Thursday August 9, 2012**  
**10:00 am to 12:30 pm**  
**170 Barrows Hall**

**Closed Books and Closed Notes**  
**Each Question is Worth 25 Points**

**Useful Formulae**

For all the corotational bases shown in the figures

$$\begin{aligned}\mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x.\end{aligned}\tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point  $P$  will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}.\tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}.\tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of  $K$  forces and a pure moment  $\mathbf{M}_p$  is

$$\dot{T} = \sum_{i=1}^K \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}.\tag{4}$$

Here,  $\mathbf{v}_i$  is the velocity vector of the point where the force  $\mathbf{F}_i$  is applied.

**Question 1**  
*A Particle in a Slotted Disk*

As shown in Figure 1, a particle of mass  $m_1$  is attached by a pair of identical springs to a circular disk of mass  $m_2$ . The particle is free to move in a smooth groove milled on the surface of the circular disk. The disk is fixed at its center point  $O$  and is free to rotate about the vertical  $\mathbf{E}_z$  axis.

The position vector of the particle relative to the center  $O$  of the disk is

$$\mathbf{x}_1 = h\mathbf{e}_y + x\mathbf{e}_x \quad (5)$$

where  $h$  is constant. The moment of inertia of the disk is  $I_{Ozz}$ , and the stiffnesses of the springs are  $\frac{K}{2}$ .

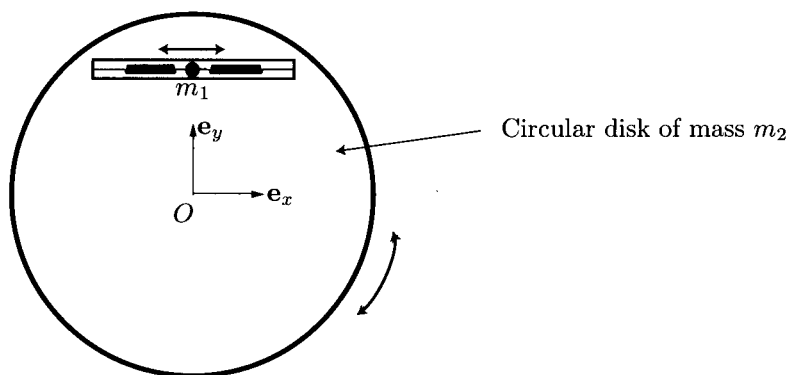


Figure 1: A particle of mass  $m_1$  is free to move in a smooth slot on a circular disk of mass  $m_2$ .

(a) (10 Points) Show that the kinetic energy and angular momentum relative to  $O$  of the particle-disk system is

$$\begin{aligned} \mathbf{H}_O &= H_O \mathbf{E}_z = \left( I_{Ozz} + m_1 (h^2 + x^2) \right) \dot{\theta} \mathbf{E}_z - m_1 h \dot{x} \mathbf{E}_z, \\ T &= \frac{m_1}{2} \dot{x}^2 + \frac{1}{2} \left( I_{Ozz} + m_1 (h^2 + x^2) \right) \dot{\theta}^2 - m_1 h \dot{x} \dot{\theta}. \end{aligned} \quad (6)$$

(b) (5 Points) Draw a free-body diagram of the particle and a free-body diagram of the circular disk.

(c) (5 Points) Show that the equations governing the motion of the disk-particle system are

$$\frac{dH_O}{dt} = 0, \quad m_1 (\ddot{x} - h\ddot{\theta} - x\dot{\theta}^2) = -Kx. \quad (7)$$

(d) (5 Points) Give an expression for the total energy  $E$  of the system and verify with the help of (7), that  $\dot{E} = 0$ .

**Question 2**  
*A Sliding Rigid Body*

As shown in Figure 2, a circular cylinder of radius  $R$ , mass  $m$ , and moment of inertia (relative to its center of mass)  $I_{zz}$  slides on a rough inclined plane. The position vector of the center of mass  $C$  has the representation

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y_0\mathbf{E}_y, \quad (8)$$

where  $y_0$  is a constant. The point  $P$  is the instantaneous point of contact.

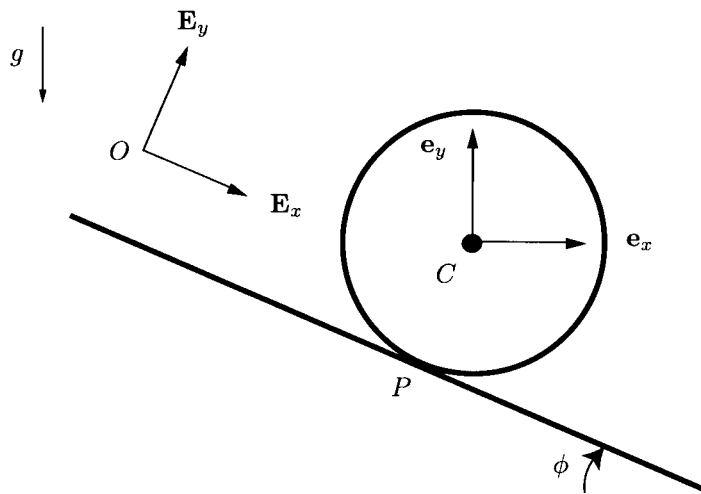


Figure 2: A rigid cylinder of mass  $m$  and radius  $R$  sliding on an inclined plane.

(a) (5 Points) Using the identity  $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$  applied to two points on the rigid body, show the velocity vector of the instantaneous point of contact has the representation

$$\mathbf{v}_P = v_P\mathbf{E}_x, \quad v_P = \dot{x} + R\dot{\theta}, \quad (9)$$

where  $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$  is the angular velocity of the rigid body.

(b) (5 Points) Draw a free-body diagram of the rigid body. In your free-body diagram give a clear expression for the friction force.

(c) (6 Points) Using balances of linear and angular momentum, show that the differential equation governing the slip velocity  $v_P$  is

$$\dot{v}_P = g \sin(\phi) - \mu_d g \cos(\phi) \left( \frac{I_{zz} + mR^2}{I_{zz}} \right) \frac{v_P}{|v_P|}, \quad (10)$$

where  $\mu_d$  is the coefficient of dynamic friction.

(d) (4 Points) Starting from the work-energy theorem (4), prove that the total energy  $E$  of the sliding rigid body decreases with time.

(e) (5 Points) Suppose  $\phi = 0$  and that the body is given an initial velocities  $\dot{x}_0$  and  $\dot{\theta}_0$  where  $v_{P_0} = \dot{x}_0 + R\dot{\theta}_0 < 0$ . Determine the time  $\tau$  that it will take for the slip velocity to reach zero.

**Question 3**  
*A Tipping Point*

As shown in Figure 3, a slender rod of mass  $m$ , length  $L$ , and moment of inertia  $I_{zz} = \frac{mL^2}{12}$  is at rest in a vertical position with the material point labelled  $A$  in contact with a rough horizontal surface. The rod is given a slight inclination and released. As the rod falls, the friction force at  $A$  eventually become insufficient to prevent the point  $A$  from sliding.

During its motion, a vertical gravitational force  $-mg\mathbf{E}_y$  acts on the rod.

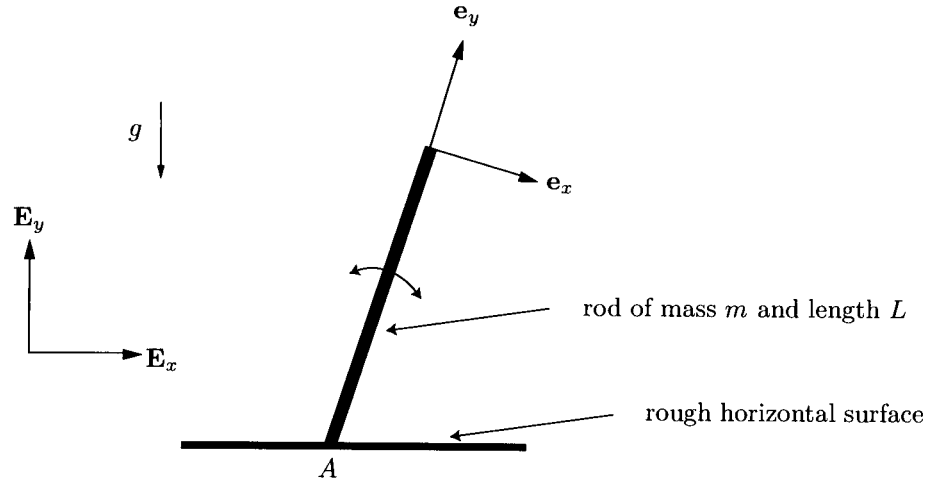


Figure 3: A slender rod of mass  $m$  is free to rotate about  $A$  and the point  $A$  is free to move on a rough horizontal surface.

The position vector of the center of mass  $C$  has the representation

$$\bar{\mathbf{x}} = \frac{L}{2}\mathbf{e}_y + \mathbf{x}_A, \quad (11)$$

where  $\mathbf{x}_A$  is the position vector of  $A$ .

(a) (5 Points) Show that the angular momentum  $\mathbf{H}_A$  of the rod and the kinetic energy of the rod have the representations

$$\mathbf{H}_A = I_{Azz}\dot{\theta}\mathbf{E}_z - \frac{mLv_A}{2}\cos(\theta)\mathbf{E}_z, \quad T = \frac{1}{2}I_{Azz}\dot{\theta}^2 + \frac{m}{2}v_A^2 - \frac{mv_AL}{2}\dot{\theta}\cos(\theta), \quad (12)$$

where  $I_{Azz} = \frac{mL^2}{3}$  and  $\dot{\mathbf{x}}_A = v_A\mathbf{E}_x$ .

(b) (5 Points) Assuming that  $A$  is in contact with the horizontal surface, draw a free-body diagram of the rod. Distinguish the cases where  $\mathbf{v}_A = \mathbf{0}$  and  $\mathbf{v}_A \neq \mathbf{0}$ .

(c) (10 Points) Assuming that  $A$  is stationary, show that

$$I_{Azz}\ddot{\theta} = \frac{mgL}{2}\sin(\theta), \quad \mathbf{R}_A = mg\mathbf{E}_y - \frac{mL}{2}(\ddot{\theta}\mathbf{e}_x + \dot{\theta}^2\mathbf{e}_y) \quad (13)$$

where  $\mathbf{R}_A$  is the reaction force at  $A$ . In addition, using the work-energy theorem (4), prove that the total energy of the rod is conserved. In your solution give a clear expression for the total energy  $E$ .

(d) (5 Points) Suppose  $A$  is sliding on the horizontal surface. Outline how you would determine the equations governing the motion of the rod.

### Question 4

#### A Particle Colliding with a Rigid Body

As shown in Figure 4, a particle of mass  $m_1$  is traveling with a velocity  $-v_0\mathbf{E}_y$  when it collides with a stationary rod. The uniform rod, which has a mass  $m_2$ , length  $L$ , and a moment of inertia relative to its center of mass of  $I_{zz}$ , is free to move on a horizontal plane. Following the collision, the particle adheres to the rod and the particle-rod move as a single rigid body on the smooth horizontal plane.

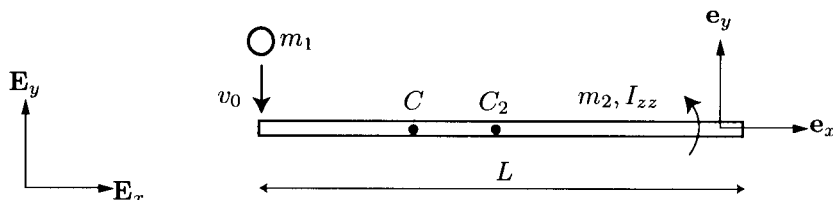


Figure 4: A particle of mass  $m_1$  collides with a rod of mass  $m_2$ . The basis vectors  $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{E}_z\}$  corotate with the rod

(a) (4 Points) Assuming that the particle adheres to the end of the rod following the collision, show that the position vector  $\bar{\mathbf{x}}$  of the center of mass  $C$  of the particle-rod following the collision has the representation

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}_2 - \left( \frac{m_1}{m_1 + m_2} \right) \frac{L}{2} \mathbf{e}_x \quad (14)$$

where  $\bar{\mathbf{x}}_2$  is the position vector of the center of mass  $C_2$  of the rod.

(b) (5 Points) Assuming that the particle adheres to the end of the rod following the collision, determine the velocity vector of the center of mass of the particle-rod immediately following the collision.

(c) (8 Points) Show that the angular velocity vector  $\boldsymbol{\omega}$  of the particle-rod immediately following the collision can be expressed in the form

$$\boldsymbol{\omega} \mathbf{E}_z = \left( \frac{m_1 m_2}{m_1 + m_2} \right) \left( \frac{L v_0}{2I} \right) \mathbf{E}_z. \quad (15)$$

For full credit, provide an expression for  $I$ .

(d) (8 Points) Show that the kinetic energy lost during the collision can be expressed in the form

$$\Delta T = \frac{1}{2} m_1 v_0^2 (1 - H). \quad (16)$$

For full credit, provide an expression for  $H$ .