Second Midterm Examination
Wednesday April 6 2011
Closed Books and Closed Notes

Question 1 *Planar Motion of a System of Two Particles* *(20 Points)*

As shown in Figure 1, a particle of mass m_1 is at rest and is attached to a fixed point O by a linear spring of stiffness K and unstretched length L_0. At time t = 0, a particle of mass m_2 traveling with a velocity vector $v_0\mathbf{E}_x$ impacts the particle of mass m_1. After the collision both particles adhere to each other, and can be considered as a particle of mass $m_1 + m_2$.

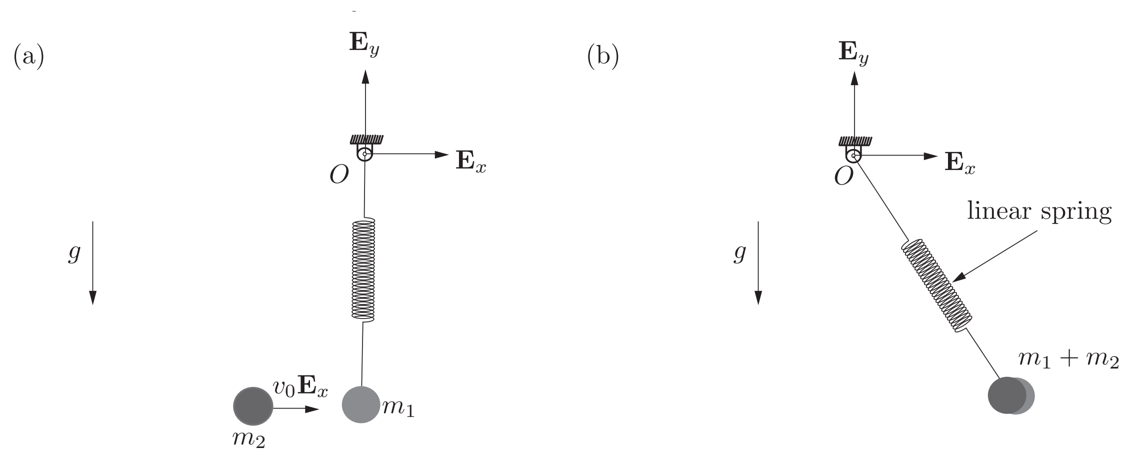

Figure 1: A system of two particles: (a) Prior to impact at $t = 0$, and (b) following the impact.

(a) *(4 Points)* Starting from the representation

$$\mathbf{r}_1 = r\mathbf{e}_r, \tag{1}$$

where $\mathbf{e}_r$ is a unit vector pointing from O along the spring to m_1, establish representations for the linear momentum $\mathbf{G}$, kinetic energy T, and acceleration $\mathbf{a}$ of the particle of mass $m_1 + m_2$ after the collision.

(b) *(4 Points)* Show that the velocities of the particle of mass $m_1 + m_2$ immediately following the collision are

$$\dot{r}(t = 0) = 0, \qquad r_0\dot{\theta}(t = 0) = \frac{m_2}{m_1 + m_2}v_0. \tag{2}$$

(c) *(4 Points)* Verify that the kinetic energy of the system is not conserved during the collision.

(d) *(4 Points)* Draw a freebody diagram of the particle of mass $m_1 + m_2$ following the collision. Give a clear expression for the spring force acting on the particle.

(e) *(4 Points)* Consider the system after impact. Starting from $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ for a single particle, show that the total energy E of the particle of mass $m_1 + m_2$ is conserved. In your solution, give a clear expression for E.

Question 2 *A Double Pendulum* *(30 Points)*

As shown in Figure 2, a mechanical system consists of two particles. The particle of mass m_2 is connected using a pin joint and a rod of length L_1 to the particle of mass m_1. The particle of mass m_1 is attached by linear spring of stiffness K and unstretched length L_0 to a fixed point O. Both particles move on a smooth vertical plane.

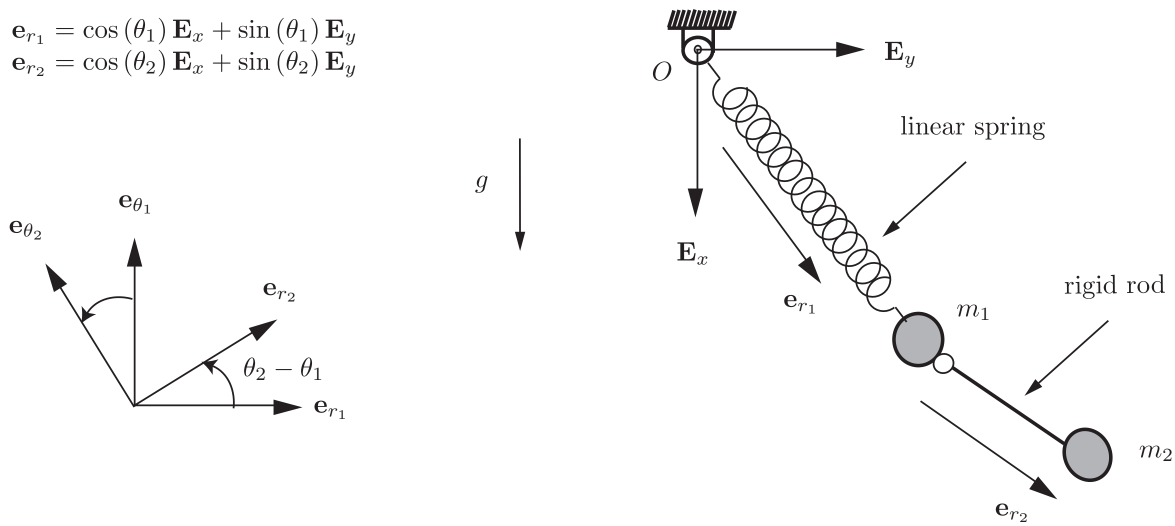

Figure 2: A system of two particles in motion on a smooth vertical plane.

(a) *(4 Points)* Starting from the representations for the position vectors of m_1 and m_2:

$$\mathbf{r}_1 = r\mathbf{e}_{r_1}, \qquad \mathbf{r}_2 = \mathbf{r}_1 + L_1\mathbf{e}_{r_2}, \tag{3}$$

establish an expression for the position vector $\mathbf{r}$ of the center of mass of the system. In addition, establish an expression for the linear momentum $\mathbf{G}$ of the system.

(b) *(8 Points)* Show that the kinetic energy of the system has the representation

$$T = \frac{m_1 + m_2}{2}\left(\dot{r}^2 + r^2\dot{\theta}_1^2\right) + \frac{m_2}{2}L_1^2\dot{\theta}_2^2 + \text{missing terms.} \tag{4}$$

For full credit supply the missing terms. (Hint: Notice the definition of $\mathbf{e}_{\theta_2}$ shown in Figure 2.)

(c) *(6 Points)* Draw 3 free-body diagrams: one for each of the individual particles and one for the system of particles. In your solution, give clear expressions for the spring force and tension force.

(d) *(7 Points)* Using the angular momentum theorem, show that $\dot{\mathbf{H}}_O \cdot \mathbf{E}_z$ depends entirely on the moments due to the gravitational forces on the particles. (Hint: use the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{b} \times \mathbf{E}_z) \cdot \mathbf{a}$).

(e) *(5 Points)* Give an expression for the total energy E of the system of particles. Then, starting from the work-energy theorem for a system of particles,

$$\dot{E} = \mathbf{F}_{nc_1} \cdot \mathbf{v}_1 + \mathbf{F}_{nc_2} \cdot \mathbf{v}_2, \tag{5}$$

show that E is conserved.