First Midterm Examination
Closed Books and Closed Notes

Question 1
A Planar Pendulum (25 Points)

As shown in Figure 1, a particle of mass m is attached to a fixed point O by a linearly elastic spring. The spring has a stiffness K and an unstretched length L. The motion of the particle is on the $\mathbf{E}_x - \mathbf{E}_y$ plane.

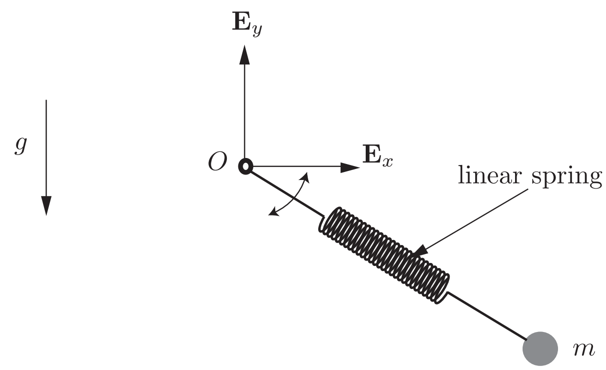

Figure 1: *Schematic of a particle of mass m which is attached to a fixed point O by a linearly elastic spring. A vertical gravitational force* $-mg\mathbf{E}_y$ *acts on the particle.*

(a) Starting from the standard representations for the position vector

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y = r\mathbf{e}_r, \tag{1}$$

establish expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle. In your solution, it is not necessary to derive the intermediate results $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$.

(b) Draw a freebody diagram of the particle. Your freebody diagram should include a normal force $N\mathbf{E}_z$ and a clear expression for the spring force.

(c) Show that the differential equations governing the motion of the particle are

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -K\left(r - L\right) - mg\sin(\theta), \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = -mg\cos(\theta). \tag{2}$$

What is the normal force acting on the particle?

(d) Show that the differential equations governing the motion of the particle can also be expressed in the form

$$\begin{aligned} m\ddot{x} &= -K\left(\sqrt{x^2+y^2} - L\right)\frac{x}{\sqrt{x^2+y^2}}, \\ m\ddot{y} &= -mg - K\left(\sqrt{x^2+y^2} - L\right)\frac{y}{\sqrt{x^2+y^2}}. \end{aligned} \tag{3}$$

(e) With the help of (2), show that it is possible for the particle to be at rest with $\theta = 270°$ and $r = \frac{mg}{K} + L$. Give a physical interpretation of this result.

Question 2

A Particle on a Helix (25 Points)

As shown in Figure 2, a bead of mass m is free to move on a rough curve in the shape of a right-handed circular helix. In addition to friction and normal forces, a vertical gravitational force $-mg\mathbf{E}_z$ acts on the bead.

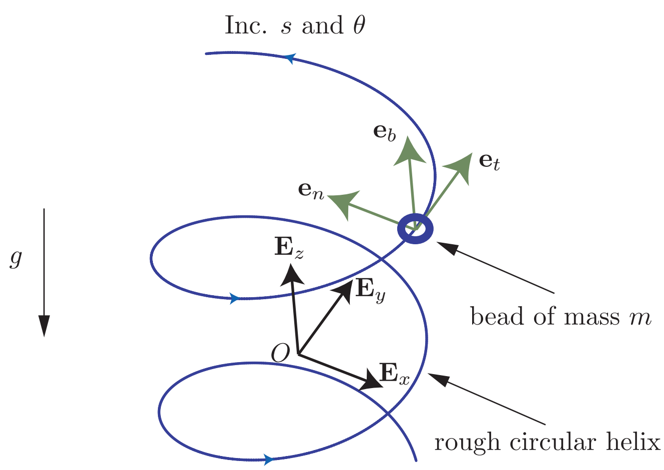

Figure 2: *A particle of mass m moving on a rough circular helix.*

(a) Using a cylindrical polar coordinate system, the position vector of a particle moving on the helix can be described as

$$\mathbf{r} = R\mathbf{e}_r + \alpha R\theta\mathbf{E}_z. \tag{4}$$

Derive expressions for the speed v, and velocity vector $\mathbf{v}$ and acceleration vector $\mathbf{a}$ vector of the particle.

(b) From your results in **(a)** and assuming that $\dot{\theta} > 0$, show that the Frenet triad for the helix is

$$\mathbf{e}_t = \frac{1}{\sqrt{1+\alpha^2}}\left(\mathbf{e}_\theta + \alpha\mathbf{E}_z\right), \qquad \mathbf{e}_n = -\mathbf{e}_r, \qquad \mathbf{e}_b = \frac{1}{\sqrt{1+\alpha^2}}\left(-\alpha\mathbf{e}_\theta + \mathbf{E}_z\right), \tag{5}$$

What is the curvature κ of the helix?

(c) Draw a freebody diagram of the particle. Give clear expressions for the forces acting on the particle, and distinguish the static friction and dynamic friction cases.

(d) Suppose that the particle is moving on the curve with $\dot{\theta} > 0$. Show that the equation governing the motion of the particle is

$$mR\sqrt{1+\alpha^2}\ddot{\theta} = -\frac{mg\alpha}{\sqrt{1+\alpha^2}} - \mu_d \left\|\mathbf{N}\right\|, \tag{6}$$

where $\mathbf{N}$ is the normal force. How would you determine $\mathbf{N}$?

(e) Suppose that the particle is stationary at a point on the helix. Show that the friction force and normal force acting on the particle are

$$\mathbf{F}_f = \frac{mg\alpha}{\sqrt{1+\alpha^2}}\mathbf{e}_t, \qquad \mathbf{N} = \frac{mg}{\sqrt{1+\alpha^2}}\mathbf{e}_b. \tag{7}$$

Show that the particle will remain stationary provided $\alpha \leq \mu_s$. Give a physical interpretation of this result.