Second Midterm Examination
Wednesday November 7 2007
Closed Books and Closed Notes

Question 1
A Single Particle (20 Points)

As shown in Figure 1, a particle of mass m is in motion about a fixed point O. A force $\mathbf{F}$ acts on the particle. This force is a central force, i.e., $\mathbf{F} \parallel \mathbf{r}$.

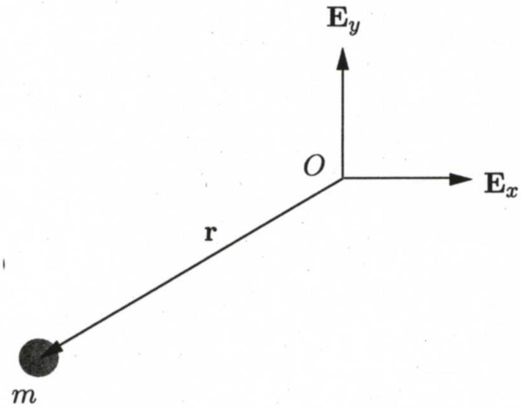

Figure 1: A particle moving on a planar path under the influence of a central force $\mathbf{F}$.

(a) *(6 Points)* Starting from the representation $\mathbf{r} = r\mathbf{e}_r$, establish expressions for the kinetic energy T and angular momentum $\mathbf{H}_O$ of the particle.

(b) *(4 Points)* Starting from the angular momentum theorem for a single particle, prove that $mr^2\dot{\theta}$ is conserved.

(c) *(5 Points)* Starting from the work-energy theorem for a single particle $\dot{T} = \mathbf{F} \cdot \mathbf{v}$, prove that the total energy of the particle is conserved if

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r. \tag{1}$$

In your solution, give a clear expression for the total energy E of the particle.

(d) *(5 Points)* A satellite is in motion in an elliptical orbit about the Earth. Show that the radial velocity $v = \dot{r}$ of the satellite varies as its distance r from the Earth:

$$v^2 = \frac{2E_0}{m} + \frac{2GM}{r} - \frac{h^2}{m^2r^2}, \tag{2}$$

where E_0 and h are constants and M is the mass of the Earth.

Question 2

A System of Two Particles (35 Points)

As shown in Figure 2, a satellite of mass m_1 is connected to a spacecraft of mass m_2 by a tether of length r. By varying the length of the tether, the rotation of the spacecraft-satellite about their center of mass C can be changed and this can then be used to artificially induce gravity.

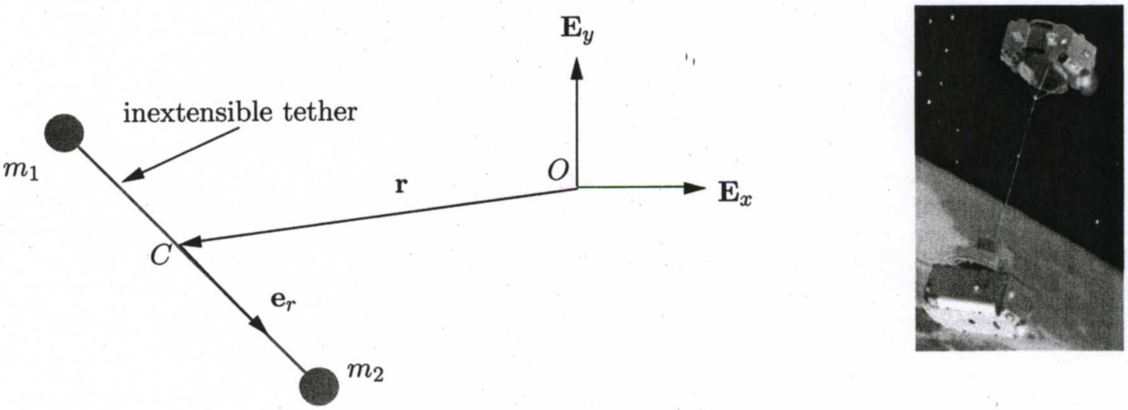

Figure 2: Schematics of a satellite and spacecraft connnected by an inextensible tether of length $r = r(t)$.

(a) *(8 Points)* Starting with the representations for the position vector $\mathbf{r}$ of the center of mass C and the position vectors of the particles

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y, \quad \mathbf{r}_1 = \mathbf{r} - r_1\mathbf{e}_r, \quad \mathbf{r}_2 = \mathbf{r} + r_2\mathbf{e}_r, \tag{3}$$

where $r = r_1 + r_2$, show that

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) r, \quad r_2 = \left(\frac{m_1}{m_2}\right) r_1, \quad m_1 r_1^2 + m_2 r_2^2 = \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2. \tag{4}$$

(b) *(10 Points)* Show that the linear momentum and angular momentum of the system are

$$\mathbf{G} = (m_1 + m_2)\,\dot{\mathbf{r}}, \quad \mathbf{H}_O = \left((m_1 + m_2)(x\dot{y} - y\dot{x}) + \left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2\dot{\theta}\right)\mathbf{E}_z. \tag{5}$$

(c) *(4 Points)* Draw free-body diagrams for each of the particles. Give a clear expression for the tension force in the tether.

(d) *(4 Points)* Show that the kinetic energy T and the linear momentum $\mathbf{G}$ of the system are constant. Clearly indicate any intermediate results that you use.

(e) *(4 Points)* Using the angular momentum theorem $\dot{\mathbf{H}}_C = (\mathbf{r}_1 - \mathbf{r}) \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}) \times \mathbf{F}_2$, show that

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right) r^2\dot{\theta} = h, \tag{6}$$

where h is a constant.

(f) *(5 Points)* Suppose an acceleration of $0.75g$ for an object on the satellite is sought. Show that this can be achieved if $\left(\frac{m_2}{m_1+m_2}\right) r\dot{\theta}^2 = 0.75g$.