CS 70	Discrete Mathematics and Probability Theory	
Fall 2010	Tse/Wagner	MT 2

Problem 1. [Rolling Dice] (25 points)

You roll a fair die three times. Consider the following events:

A = first roll is a 3
B = second roll is a 4
C = second roll is a 3
D = the first roll is a 3 and the second roll is a 4
E = the sum of the first and second roll is 5
F = the sum of the first and second roll is 3
G = the sum of the second and third roll is 6
H = the first roll is different from the second roll
I = the first roll is different from the third roll

(a) A and B are disjoint.

(b) A and C are disjoint.

(c) $\Pr[A] < \Pr[D]$

(d) A and E are independent.

(e) A and F are independent.

(f) A and H are independent.

(g) H and I are disjoint.

(h) H and I are independent.

(i) (2 points) Compute $\Pr[E \cap G]$.

(j) (3 points) Compute $\Pr[B|G]$.

(k) (4 points) Let X be the value of the first roll and Y be the value of the second roll. Compute and plot the distribution of $Z = \min(X,Y)$, i.e., Z is the minimum of X and Y.

Problem 2. [Trapping Ants] (27 points)

An ant starts in the lower-left corner of the following grid (i.e., at $S = (0,0)$) and wants to get to the upper-right corner $F = (4,4)$:

				F
S				

The ant moves from one cell to an adjacent cell subject to the following restrictions: the ant only moves up or right (never left, down, or diagonal), and the ant never goes outside the grid. Answer the following questions. Unless specified, you do not need to show your work.

(a) (3 points) Consider sequences of the letters R and U, where R means that the ant moves right and U means that the ant moves up. Give an example of one such sequence that corresponds to a path from S to F. How many paths can the ant take to get from S to F?

(b) (3 points) Now, assume there is a single ant trap, located in the center of the grid, at cell $(2,2)$. How many paths are dangerous, i.e., how many paths from S to F go through the trap?

(c) (3 points) More generally, suppose there is a single ant trap, in the cell $(k,4-k)$, where $k \in \{0,1,2,3,4\}$, i.e., in *one* of the cells marked T in the figure below:

T				F
	T			
		T		
			T	
S				T

How many paths are now dangerous? (Your answer should be a function of k.)

(d) (4 points) Now the ant picks a path from S to F uniformly at random from all possible paths and a human picks one of the T cells uniformly at random and puts a trap there. The ant's path and the location of the ant's trap are independently chosen. Using your answers to parts (a) and (c), give an expression for the probability that the ant gets trapped. Leave your answer in terms of a single summation. Show your work.

(e) Now we calculate the probability in part (d) in another way.

(i) (3 points) Let $P_1, P_2, \ldots, P_n$ be the paths that the ant can take from S to F. Consider an arbitrary path P_i. How many times can it pass through a cell marked T? Does your answer depend on the path?

(ii) (4 points) Using your answer to (e)(i) or otherwise, determine the probability that the ant gets trapped conditional on it choosing a particular path P_i. The answer should be a number.

(iii) (4 points) Using your answer to part (e)(ii), compute the probability that an ant's randomly selected path goes through the randomly selected trap. The answer should be a number and should not involve summations. Show your work.

(iv) (3 points) Suppose now the ant does not choose a path uniformly at random but the human still puts the trap uniformly at random on one of the T cells. Does the answer to part (e)(iii) change? Why?

Problem 3. [Intelligent Guessing] (20 points)

Your friend has a bag with two coins, one of which is a fair coin and the other a trick coin with Heads on both sides. The game is that he picks one of the coins randomly and flips it several times, and you guess which coin he picked based on the outcome of the flips.

(a) (3 points) Your friend picks a coin from the bag uniformly at random and flips it once, resulting in a Heads. Which coin would you guess? What is the probability that you are wrong? Show your work.

(b) (2 points) Your friend picks a coin from the bag uniformly at random and flips it twice, resulting in Heads once and Tails once. Which coin would you guess? What is the probability that you are wrong? Show your work.

(c) (5 points) Your friend picks a coin from the bag uniformly at random and flips it twice, resulting in two Heads. Which coin would you guess? What is the probability that you are wrong? Show your work.

(d) (5 points) Now suppose your friend did not pick uniformly at random from the two coins but instead picks the fair coin with probability p and the trick coin with probability $1-p$. What is the range of values of p for which you will change the guess you made in part (c)? Show your work.

(e) (5 points) Now suppose you are conservative and are not willing to guess unless you have less than a 10% chance of making a mistake. Assuming that your friend picks one of the two coins uniformly at random, how many Heads do you have to observe in a row before you are willing to guess? Show your work.

Problem 4. [Weather Forecasting] (28 points)

The weather forecast says that each of the next 7 days has a 50% chance of rain and 50% chance of sun. Each day is either rainy or sunny (those are the only two possibilities, and both are equally likely), and the weather on any day is independent of all the other days.

Define the events $R_1, \ldots, R_7$ as follows: R_i is the event that it is rainy on the ith day. Let the random variable X denote the total number of days that it rains, out of the next 7 days.

(a) (1 point) Are the events R_1, R_2 independent? You don't need to justify your answer.

(b) (1 point) Are the events $R_1, R_2, \ldots, R_7$ mutually independent? You don't need to justify your answer.

(c) (4 points) Calculate the following probabilities. You do not need to show your work.

$\Pr[X = 0] =$ $\qquad$ $\Pr[X = 1] =$

$\Pr[X = 2] =$ $\qquad$ $\Pr[X = 3] =$

$\Pr[X = 4] =$ $\qquad$ $\Pr[X = 5] =$

$\Pr[X = 6] =$ $\qquad$ $\Pr[X = 7] =$

(d) (4 points) Calculate the expectation of X. Show your work. Circle your final answer. Your final answer should be a number (not an unevaluated expression).

The weather service cancels the previous forecast and issues a new weather forecast. The new weather forecast says that there is a 50% chance of rain and a 50% chance of sun on the first day; also, on each subsequent day, there is a $\frac{2}{3}$ chance that the weather is the same as the previous day, and a $\frac{1}{3}$ chance that the weather is the opposite of what it was on the previous day. Each day is either rainy or sunny (those are the only two possibilities).

(e) (4 points) With the new forecast, calculate the following probabilities. You do not need to show your work.

$\Pr[R_1] =$ $\qquad$ $\Pr[R_2] =$

$\Pr[R_3] =$ $\qquad$ $\Pr[R_4] =$

$\Pr[R_5] =$ $\qquad$ $\Pr[R_6] =$

$\Pr[R_7] =$

(f) (2 points) With the new weather forecast, are the events R_1, R_2 independent? Why?

(g) (2 points) With the new weather forecast, are the events $R_1, R_2, \ldots, R_7$ mutually independent? Why?

(h) (5 points) With the new forecast, calculate the following probabilities. You do not need to show your work.

$$\Pr[X = 0] =$$

$$\Pr[X = 1] =$$

(i) (4 points) With the new weather forecast, calculate the expectation of X. Show your work. Circle your final answer. Your final answer should be a number (not an unevaluated expression).

(j) (1 point) Is your answer to part (i) larger than, the same as, or smaller than your answer to part (d)? You don't need to justify your answer.