3 Problem 1

(i) A hot cup of coffee does decrease in mass as it cools down. It loses energy through radiating that heat away.

(i) Event A happened before Event B in a certain frame, and that the two events are spacelike separated. This means there exists a certain frame where B happens before A.

Because they are spacelike separated, we can boost to a frame where A and B occur at the same time. The x' axis of that frame will connect the dots A and B. Boosting a little more will cause B to happen before A.

A happening at the same position as B in some frame would say the invariant interval would be $\Delta s^2 = c^2\Delta t^2 - 0^2 \geq 0$ which is time-like. We can't get from space-like to time-like with changing reference frame.

A photon passing from A to B would be travelling at c so the invariant interval on that trajectory would be $c^2\Delta t^2 - (c\Delta t)^2 = 0$. Again that is impossible with a change in reference frame.

(i) If you have a 30 W blue lamp and a 30 W red lamp, the red lamp gives off more photons per second. We know how much energy needs to be outputed per second, 30 J. That has to be split up among all the photons. Blue light has higher energy per photon, so fewer are needed to acheive that same energy.

(i) Wien's law says $\lambda_{max}T = 2.898 \times 10^{-3}mK$. Therefore changing the temperature does change the most common wavelength emitted.

(i) In the photoelectric effect, changing only the intensity of the light only effects the number of photo-electrons ejected. It does not increase the energy of each ejected electrons. This is because there are more photons, but each photon gives the same kick to each electron it hits.

(i) Comparing two reference frames S and S' where S' is moving in the +x direction relative to S. The momentum in the y direction is the same in both frames. This is because looking at the formula for how energy and momentum transform we see that energy and x momentum will mix up under this change of frame and leave the other directions, in particular y, alone. The momentum in the y direction may be the same but γ is different so that means v_y is different.

4 Problem 2

Kirk boards his starship, Enterprise, and launches from Earth to Vulcan, travelling at $\frac{3}{5}c$ while his friend McCoy stays behind on Earth. Immediately after Kirk launches (in Kirk's frame), Spock launches a shuttle from Vulcan going to Earth at $\frac{4}{5}c$. The distance from Earth to Vulcan is 24 light years as measured in the rest frame of the two planets.

4.1 Part a

If Kirk launches at $t = 0$ in McCoy's frame, at what time does Spock launch according to McCoy.

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-.6^2}} = \frac{1}{.8} \\
t'_K &= t'_S \\
t_K &= 0 \\
t_K &= \gamma(t'_K + \frac{v}{c^2}x'_K) \\
t'_K &= 0 \\
t_S &= \gamma(t'_S + \frac{v}{c^2}x'_S) \\
&= \gamma(0 + \frac{v}{c^2\gamma}(24l.y.)) \\
&= \frac{1}{.8}.8\frac{3}{5}24years \\
&= 14.4yrs
\end{aligned}
$$

4.2 Part b

As Kirk passes Spock, how long does Spock's shuttle appear to Kirk, if the shuttle's rest length is 20 m.

Spock appears to be moving with a larger velocity given by the velocity addition formula.

$$
\begin{aligned}
v &= \frac{v_K + v_S}{1 + v_K v_S/c^2} \\
&= \frac{3/5 + 4/5}{1 + 12/25}c \\
&= \frac{35c}{37} \approx 0.945946c \\
\gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{37}{12} \\
L' &= \frac{L}{\gamma} = \frac{20m * 12}{37} = \frac{240}{37}m \approx 6.48649m
\end{aligned}
$$

4.3 Part c

How long does the trip take Kirk in his own reference frame, T'_K? How long does the trip take Spock in his own reference frame T''_S?

Kirk sees Vulcan coming to him at $\frac{3}{5}c$ and sees an initial distance between them as $\frac{24ly}{\gamma}$. Therefore, he takes $\frac{24ly}{\gamma * \frac{3}{5}c} = 32yrs$

Spock sees Earth coming to him at $\frac{4}{5}c$ and sees an initial distance between them as $\frac{24ly}{\gamma_{Spock}}$. This is the γ associated with Spock's velocity.

$$
\begin{aligned}
\gamma_{Spock} &= \frac{1}{\sqrt{1 - .8^2}} = \frac{1}{.6} \\
t''_S &= \frac{24ly * .6}{4/5c} = 18yrs
\end{aligned}
$$

4.4 Part d

Who reaches their destination first according to McCoy? Who reaches their destination first according to Kirk?

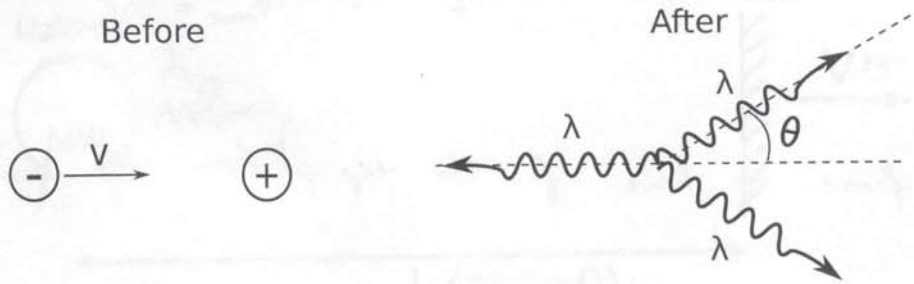
According to McCoy, Spock takes $\frac{24ly}{\frac{4}{5}c} = 30yrs$ and Kirk takes $\frac{24ly}{\frac{3}{5}c} = 40yrs$. But McCoy also thinks Kirk had a head start of $14.4yrs$ which is enough for Kirk to reach Vulcan before Spock reaches Earth.

According to Kirk, he reaches Vulcan at $T'_K = 32yrs$ and we need to find T'_S. We do this by transforming the event of Spock arriving on Earth between McCoy's and Kirk's frame.

$$
\begin{aligned}
T'_S &= \gamma(T_S - \frac{v}{c^2} X_S) \\
T_S &= 30yrs + 14.4yrs = 44.4yrs \\
X_S &= 0 \\
T'_S &= \frac{1}{.8} 44.4yrs = 55.5yrs
\end{aligned}
$$

So Kirk reaches first in his frame too.

Problem 3 [24 pts]



An electron (mass m) moving with speed v collides with a stationary positron. The two annihilate and create three photons with the same wavelength.

a) [8 *pts*] What is the wavelength of the three photons?

b) [6 *pts*] At what angle θ does the top photon leave at?

c) [4 *pts*] What is the max speed v_{max}, beyond which this reaction cannot occur? *Hint: A useful identity in this problem is:* $\frac{v}{c} = \frac{\sqrt{(\gamma+1)(\gamma-1)}}{\gamma}$.

d) [6 *pts*] In rest frame of the positron, at what angle θ' does the top photon leave at now, in terms of θ, γ, λ and v? *Remember,* $\tan\theta = \frac{p_y}{p_x}$.

a) initially $E_{tot} = \gamma(v) m c^2 + \gamma(0) m c^2 = (1+\gamma(v)) m c^2$. 2pt w/. $\gamma(x) = \frac{1}{\sqrt{1+\frac{x^2}{c^2}}}$

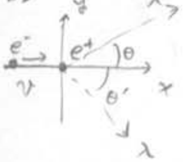
||

finally $E_{tot} = 3 \cdot E_{photon}(\lambda) = 3 \cdot \frac{hc}{\lambda}$ — 2pt

$\Rightarrow \lambda = \frac{3h}{(1+\gamma(v))mc}$ — 2pt

w/ $E_{photon} = |\mathbb{P}_{photon}| \cdot c \Rightarrow |\mathbb{P}_{photon}| = \frac{(1+\gamma(v))}{3} mc$ — 2pt

b)



initially. $P^y_{tot} = 0$ $\qquad P^x_{tot} = \gamma(v) m v$

||

finally $P^y_{tot} = |\mathbb{P}_{photon}| \cdot (\sin\theta - \sin\theta')$. $\qquad P^x_{tot} = |\mathbb{P}_{photon}| \cdot (\cos\theta + \cos\theta' - 1)$ 2pt.

y-component $\Rightarrow$. $\sin\theta = \sin\theta' \Rightarrow \theta = \theta'$

x-component $\Rightarrow$. $|P_{photon}| \cdot (2\cos\theta - 1) = \gamma(v) m v$. 2pt

$\therefore \cos\theta = \frac{1}{2}\left\{\frac{\gamma(v) m v}{|P_{photon}|} + 1\right\} = \frac{1}{2}\left\{1 + \frac{3\gamma(v)}{1+\gamma(v)} \cdot \frac{v}{c}\right\} = \frac{1}{2}\left(1 + 3\sqrt{\frac{\gamma(v)-1}{\gamma(v)+1}}\right)$.

or $\theta = \cos^{-1}\left\{\frac{1}{2}\left(1 + 3\sqrt{\frac{\gamma(v)-1}{\gamma(v)+1}}\right)\right\}$ 2pt.

c). since $|\cos\theta| \leq 1$. we have (since part (b) shows our $\cos\theta \geq 0$).

$$\frac{1}{2}\left(1+3\sqrt{\frac{\gamma(v)-1}{\gamma(v)+1}}\right) \leq 1$$

$$\Rightarrow \sqrt{\frac{\gamma(v)-1}{\gamma(v)+1}} \leq \frac{1}{3}$$ solve for $\gamma(v) \leq \frac{5}{4}$. or $v_{max} = \frac{3}{5}c$

— 2pt — 2pt.

d). in the original frame we have for the top photon the four-momentum.

$p^0 = |\mathbb{P}_{photon}|$

$p^1 = |\mathbb{P}_{photon}| \cdot \cos\theta$

$p^2 = |\mathbb{P}_{photon}| \cdot \sin\theta$

while the electron's rest frame moves at velocity v towards the right. we have the Lorentz transformation. 2pt.

$$p^{1\prime} = \gamma(v)\left(p^1 - \frac{v}{c}p^0\right) = \gamma(v)\cdot\left(\cos\theta - \frac{v}{c}\right)|\mathbb{P}_{photon}|$$

$$p^{2\prime} = p^2 = |\mathbb{P}_{photon}| \cdot \sin\theta$$ 2pt.

$$\Rightarrow \tan\theta' = \frac{p^{2\prime}}{p^{1\prime}} = \frac{\sin\theta}{\gamma(v)\,(\cos\theta - \frac{v}{c})} \quad \text{or} \quad \theta' = \tan^{-1}\left(\frac{\sin\theta}{\gamma(v)\,(\cos\theta - v/c)}\right)$$ — 2pt

$$\frac{1-\cos^2\theta'}{\cos^2\theta'} = \frac{1-\cos^2\theta}{\frac{1}{1-\beta^2}(\cos^2\theta - 2\beta\cos\theta + \beta^2)}.$$

$$\Rightarrow (1-\cos^2\theta')\cdot\frac{1}{1-\beta^2}\cdot(\cos\theta-\beta)^2 = (1-\cos^2\theta)\cos^2\theta'$$

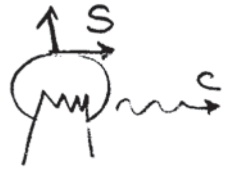
$$\Rightarrow \left(1-\cos^2\theta + \frac{(\cos\theta-\beta)^2}{1-\beta^2}\right)\cdot\cos^2\theta' = \frac{(\cos\theta-\beta)^2}{1-\beta^2}$$

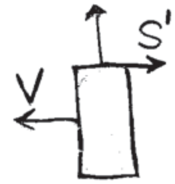
$$\Rightarrow \cos^2\theta' = \frac{(\cos\theta-\beta)^2}{\cancel{\cos^2\theta} - 2\beta\cos\theta + \cancel{\beta^2} + \beta^2\cos^2\theta - \cancel{\beta^2} + 1 - \cancel{\cos^2\theta}} = \frac{(\cos\theta-\beta)^2}{\beta^2\cos^2\theta - 2\beta\cos\theta + 1} = \frac{(\cos\theta-\beta)^2}{(\beta\cos\theta - 1)^2}$$

$$\therefore \cos\theta' = \pm\frac{\cos\theta-\beta}{\beta\cos\theta-1} \quad \text{or} \quad \pm\beta\cos\theta'\cos\theta \mp \cos\theta' = \cos\theta - \beta$$

$$\Rightarrow \cos\theta = \frac{\beta \mp \cos\theta'}{1 \mp \beta\cos\theta'}$$

$\boxed{5}$





$\boxed{\beta \equiv \frac{V}{c}}$

(a) glass appears Lorentz contracted: $\boxed{l = \frac{l_0}{\gamma}}$ $\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$

(b) Note: speed of light inside glass can be different from 'c' for both S and S', since light is not travell[ing] in vacuum

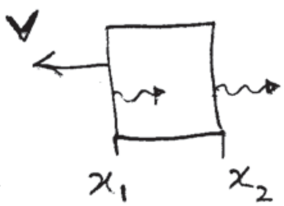
$$v = \frac{v' - V}{1 - v'V/c^2}$$

$v' = \frac{c}{n}$ as measured by S'

v as measured by S

$$= \frac{\frac{c}{n} - V}{1 - \frac{V}{c \cdot n}} \Rightarrow \boxed{v = \frac{\frac{c}{n}\left(1 - n\frac{V}{c}\right)}{1 - \frac{V}{cn}} = \frac{\frac{c}{n}(1-n\beta)}{1-\beta/n}}$$

(c)



Let photon enter glass at (x_1, t_1) and exit at (x_2, t_2)

$$x_2(t) = x_1 + \frac{l_0}{\gamma} - V(t - t_1)$$

photon coord $\leftarrow x_p(t) = x_1 + v(t - t_1)$

At t_2: $x_2(t_2) = x_p(t_2)$

$$\Rightarrow \quad x_1 + \frac{l_0}{\gamma} - V(t_2 - t_1) = \cancel{x_1} + v(t_2 - t_1)$$

$$\boxed{t_2 - t_1 = \text{transit time} = \frac{l_0/\gamma}{v + V} = \frac{l}{v + V}}$$

as expected $\longrightarrow$

(d) From (b): $v = \frac{\frac{c}{n}(1-n\beta)}{1-\beta/n}$

For photon to appear backwards as seen from S':
require $v < 0$

$$\Rightarrow \left.\begin{array}{c} (1-n\beta) < 0 \text{ and } (1-\beta/n) > 0 \\ \text{OR} \\ (1-n\beta) > 0 \text{ and } (1-\beta/n) < 0 \end{array}\right\} \Rightarrow \begin{array}{l} \text{(A)}\ \boxed{\beta > \frac{1}{n} \text{ and } \beta < n} \\ \\ \text{(B)}\ \boxed{\beta < \frac{1}{n} \text{ and } \beta > n} \end{array}$$

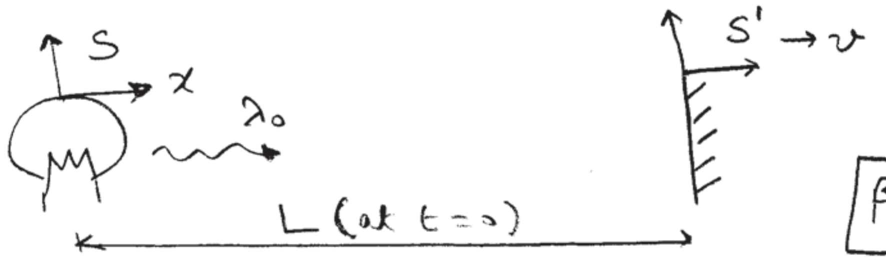
Case I: $n > 1$ (usual) $\Rightarrow$ (B) not possible $\because$ β must < 1
But (A) possible

$\therefore$ Can appear to travel backwards if $\beta > \frac{1}{n}$
e.g. glass $n \approx \frac{3}{2}$ $\Rightarrow$ need $V > \frac{2}{3}c$

Case 2: $n < 1$
(A) Not possible $\because$ β can't be $> \frac{1}{n}$
(B) Possible

$\therefore$ Can appear to travel backwards if $\beta > n$

$\boxed{4}$



$\boxed{\beta \equiv \frac{v}{c}}$

(a) As measured from S:

x coord. of mirror: $\boxed{x(t) = L + vt}$; light travels a distance 'ct' to strike the mirror at $t = T$

$\therefore \quad cT = L + vT \quad \Rightarrow \quad T = \frac{L}{c - v} = \frac{L}{c(1-\beta)}$

So, it strikes the mirror at $(x = cT,\ t = T)$. It then returns travelling the same distance cT

$$\therefore \quad \boxed{\text{Roundtrip time} = 2T = \frac{2L}{c - v} = \frac{2L}{c(1-\beta)}}$$

(b) In S', source S appears to recede – move in $-\hat{x}$ with speed v

$\therefore$ appears (redshifted) Doppler shifted.

$$\boxed{\lambda' = \lambda_0 \sqrt{\frac{1+\beta}{1-\beta}} = \lambda_0 \sqrt{\frac{1 + v/c}{1 - v/c}} = \lambda_0 \sqrt{\frac{c + v}{c - v}}}$$

[Aside: This easily follows from Lorentz transf. of $(\omega, \vec{k})$: $\quad k'_x = \gamma(k_x - \beta\,\omega/c)$ with $k_x = k = 2\pi/\lambda$, $\omega = ck$]

→

(c) To S', the reflected light appears to come from a source moving to the right with speed v emitting λ' in its rest frame.

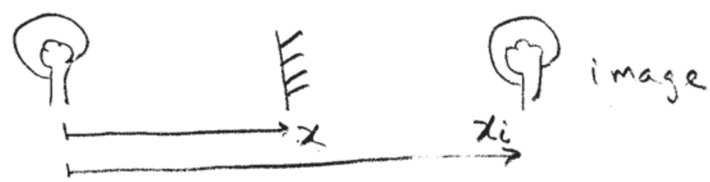
Thus, it appears redshifted

$$\lambda = \lambda' \sqrt{\frac{1+\beta}{1-\beta}} = \lambda_0 \sqrt{\frac{1+\beta}{1-\beta}} \cdot \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\Rightarrow \boxed{\lambda = \lambda_0 \left(\frac{1+\beta}{1-\beta}\right)} \qquad \frac{\lambda}{\lambda_0} = \frac{1+\beta}{1-\beta} \Rightarrow \frac{\lambda-\lambda_0}{\lambda+\lambda_0} =$$

$$\Rightarrow \beta = \frac{\lambda-\lambda_0}{\lambda+\lambda_0} \longrightarrow \boxed{v = c\left(\frac{\lambda-\lambda_0}{\lambda+\lambda_0}\right)}$$

Note: The doppler shift is not based on the speed of the image



$x = L + vt$ (from (a))

$$x_i = 2L + 2vt \Rightarrow \boxed{v_i \text{ image speed} = \frac{dx_i}{dt} = 2v}$$

$$\Rightarrow \beta_i = 2\beta$$

In fact β_i can be >1 leading to

$$\sqrt{\frac{1+\beta_i}{1-\beta_i}} = \sqrt{\frac{1+2\beta}{1-2\beta}} \quad \text{imaginary}$$

ALTERNATE SOLUTION FOR 5 (b) & 5 (c):

In S', the coordinate separation between entering & exit of photon

$$\Delta x' = l_0, \qquad \Delta t' = \frac{\Delta x'}{v'} = \frac{n l_0}{c}$$

Transform to S:

$$\Delta x = \gamma(\Delta x' - \beta c \Delta t') = \gamma\left(l_0 - \beta c \frac{n l_0}{c}\right)$$

$$\Delta t = \gamma(\Delta t' - \beta \Delta x'/c) = \gamma\left(\frac{n l_0}{c} - \frac{\beta l_0}{c}\right)$$

$$\left.\begin{array}{l} \Delta x = \gamma l_0 (1-\beta n) \\ \boxed{\Delta t = \gamma \frac{l_0}{c}(n-\beta)} \end{array}\right\} \Rightarrow \boxed{v = \frac{\Delta x}{\Delta t} = \frac{c(1-\beta n)}{n-\beta} = \frac{c}{n}\left(\frac{1-\beta n}{1-\beta/n}\right)}$$

Matches Method I

Show that Δt matches Method I result:

Method I:

$$\Delta t = \frac{l_0}{\gamma(v+V)} = \frac{l_0}{\gamma\left[\frac{c}{n}\frac{(1-n\beta)}{(1-\beta/n)} + \beta c\right]} = \frac{l_0}{\gamma c\left[\frac{1-n\beta}{n-\beta} + \beta\right]}$$

$$= \frac{l_0 (n-\beta)}{\gamma c\left[1 - \cancel{n\beta} + \cancel{n\beta} - \beta^2\right]} = \frac{l_0(n-\beta)}{\gamma c \frac{1}{\gamma^2}} \Longrightarrow \boxed{\Delta t = \frac{\gamma l_0 (n-\beta)}{c}}$$

Matches above

$\longrightarrow$