Midterm Solutions—March 03, 2005

1. (15 pts) Find a matrix X which satisfies the given conditions if possible. If not, explain why not.

   (a) (3 pts) $2X + \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

   (b) (3 pts) $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 2 & 5 & 1 \end{pmatrix}$.

   (c) (3 pts) $X = \begin{pmatrix} 3 & -2 & 0 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

   (d) (2 pts) $X \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \end{pmatrix}$, with X invertible.

   (e) (2 pts) $X \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$, with X invertible.

   (f) (2 pts) $X \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$, with X invertible.

2. (15 pts) Let

$$A := \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Use Gauss elimination in the standard way to:

(a) (5 pts) Find a basis for the row space of A.

(b) (5 pts) Find a basis for the column space of A from among the columns of A.

(c) (5 pts) Find a basis for the null space of A.

3. (20 pts) Let P_3 denote the vector space of polynomials p of degree at most three. You may assume that this is a vector space of dimension 4.

   (a) (5 pts) Prove that $(1, x^2, x^3 - x)$ is a linearly independent sequence in P_3.

   (b) (5 pts) Prove that the set W of all $p \in P_3$ such that $p(1) = p(-1)$ is a linear subspace of P_3 and that its dimension at most 3. Hint: Use the fact that $W \neq P_3$.

   (c) (5 pts) Prove that $(1, x^2, x^3 - x)$ is an ordered basis for W.

   (d) (5 pts) Find the coordinates of $(x - 1)(x + 1)$ with respect to this ordered basis.

4. (10 pts) Let $A := \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) (5 pts) Find A^{-1}.

(b) (5 pts) Write $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as a linear combination of the columns of A.

5. (10 pts) Let A be a 7×13 matrix.

(a) (5 pts) What is the maximum possible dimension of the column space of A? If this is achieved, what are the dimensions of the row and null spaces of A? Explain.

(b) (5 pts) Answer the same questions for a 13×7 matrix.