Physics 7B

Fall 07, Midterm 2

Professor Lee

Solutions

11/06/07

Prof. Lee

$\boxed{1}$

ⓐ $Q_{1,i} = C_1 V_{1,i}$; $Q_{2,i} = C_2 V_{2,i}$

$$V_{1,f} = V_{2,f} \equiv V_f$$

$$\Rightarrow \frac{Q_{1,f}}{C_1} = \frac{Q_{2,f}}{C_2} \Rightarrow Q_{2,f} = Q_{1,f} \frac{C_2}{C_1}$$

$$Q_{1,f} + Q_{2,f} = Q_{1,i} + Q_{2,i} = C_1 V_{1,i} + C_2 V_{2,i}$$

$$\Rightarrow Q_{1,f} \left(1 + \frac{C_2}{C_1} \right) = C_1 V_{1,i} + C_2 V_{2,i}$$

$$\therefore Q_{1,f} = \frac{C_1 V_{1,i} + C_2 V_{2,i}}{1 + \frac{C_2}{C_1}}$$

<u>and</u> $$Q_{2,f} = \frac{C_1 V_{1,i} + C_2 V_{2,i}}{1 + \frac{C_1}{C_2}}$$

$$V_f = \frac{Q_{1,f}}{C_1} = \frac{C_1 V_{1,i} + C_2 V_{2,i}}{C_1 + C_2}$$

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(b) $Q_{1,i} = C_1 V_{1,i}$; $Q_{2,i} = C_2 V_{2,i}$

$V_{1,f} = V_{2,f} \equiv V_f$

$\Rightarrow Q_{2,f} = Q_{1,f} \dfrac{C_2}{C_1}$

$Q_{1,f} + Q_{2,f} = Q_{1,i} - Q_{2,i} = C_1 V_{1,i} - C_2 V_{2,i}$

$\therefore Q_{1,f} = \dfrac{C_1 V_{1,i} - C_2 V_{2,i}}{1 + \frac{C_2}{C_1}}$

and $Q_{2,f} = \dfrac{C_1 V_{1,i} - C_2 V_{2,i}}{1 + \frac{C_1}{C_2}}$

$V_f = \dfrac{Q_{1,f}}{C_1} = \dfrac{C_1 V_{1,i} - C_2 V_{2,i}}{C_1 + C_2}$

Lee MT2

2) a) Calculate using $C = \frac{Q}{V}$

For $b < r < c$, we can use the voltage due to a point charge so

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

Thus, the voltage difference is

$$\Delta V = V(b) - V(c)$$

$$= \frac{Q}{4\pi\varepsilon_0 b} - \frac{Q}{4\pi\varepsilon_0 c} = \frac{Qc}{4\pi\varepsilon_0 bc} - \frac{Qb}{4\pi\varepsilon_0 bc} = \frac{c-b}{4\pi\varepsilon_0 bc} Q$$

Thus

$$C = \frac{Q}{V} = \frac{1}{\frac{c-b}{4\pi\varepsilon_0 bc}} = \boxed{\frac{4\pi\varepsilon_0 bc}{c-b}}$$

b) On a: $\vec{E}$ in shell ab is 0. Putting a Gaussian surface inside the shell gives

$$0 = \iint \vec{E}\, da = \frac{Q_{enc}}{\varepsilon_0}$$

Thus $Q_{enc} = 0$ and $\boxed{\sigma_a = 0}$

Around shell ab, we have

$$\iint E \cdot da = \frac{+Q}{\varepsilon_0}$$

All of this charge is on surface b, as $\sigma_a = 0$, so

$$\boxed{\sigma_b = \frac{+Q}{4\pi b^2}}$$

In shell cd $\vec{E} = 0$, so

$$0 = \iint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Thus on surface c, we have $Q_c = -Q$ and thus

$$\boxed{\sigma_c = \frac{-Q}{4\pi c^2}}$$

Outside d, total enclosed charge is 0. As this is accounted for by surface c + b, we have

$$\boxed{\sigma_d = 0}$$

c) as

$$W = \int_{\infty}^{0} \vec{E}\, dl$$

we only do work moving the charge through areas where $\vec{E} \neq 0$. This only occurs between b and c. However, $W = \Delta U = q\Delta V$ which for the path from c to b we already calculated in part a) giving

$$W = q\left(\frac{Q}{4\pi\varepsilon_0 b} - \frac{Q}{4\pi\varepsilon_0 c}\right)$$

$$= \frac{qQ}{4\pi\varepsilon_0}\frac{c-b}{bc}$$

d) Effectively we now have 2 capacitors hooked up in series. Without the dielectric, these would have capacitances, by analogy with a)

$$C_1 = 4\pi\varepsilon_0 \frac{b\frac{c+b}{2}}{\frac{c+b}{2}-b} = 4\pi\varepsilon_0 \frac{b(c+b)}{c-b} = 4\pi\varepsilon_0 b \frac{c+b}{c-b}$$

$$C_2 = 4\pi\varepsilon_0 \frac{\frac{c+b}{2}c}{c-\frac{c+b}{2}} = 4\pi\varepsilon_0 \frac{c(c+b)}{c-b} = 4\pi\varepsilon_0 c \frac{c+b}{c-b}$$

Filling the first capacitor with a dielectric we get $C_1 \to kC_1$. The total capacitance is now

$$\frac{1}{C} = \frac{1}{kC_1} + \frac{1}{C_2} = \frac{1}{k4\pi\varepsilon_0 b}\frac{c-b}{c+b} + \frac{1}{4\pi\varepsilon_0 c}\frac{c-b}{c+b}$$

$$= \left(\frac{c}{4\pi\varepsilon_0 kbc} + \frac{kb}{4\pi\varepsilon_0 kbc}\right)\frac{c-b}{c+b}$$

$$= \frac{kb+c}{4\pi\varepsilon_0 kbc}\frac{c-b}{c+b}$$

$$\Rightarrow C = 4\pi k\varepsilon_0 \frac{bc(c+b)}{(c-b)(c+kb)}$$

Inside a dielectric the electric field behaves as $E \to \frac{E}{k}$. Using the expression for V from a)

$$E = -\frac{\partial}{\partial r}V = -\frac{Q}{4\pi\varepsilon_0 r^2} \rightarrow E' = -\frac{Q}{4\pi k\varepsilon_0 r^2}$$

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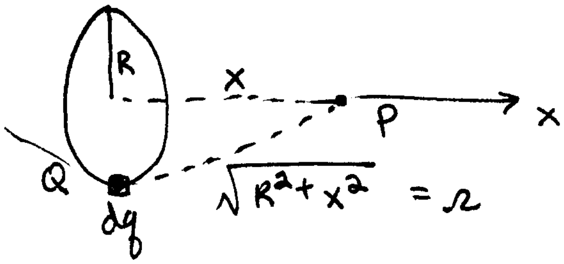
Solutions for #3

Michelle Yong

a)



r = dist. from dq to point P

use $V = \int \frac{dq}{4\pi\epsilon_0 r}$

$dq = \lambda\, dl = \frac{Q R\, d\varphi}{2\pi R} = \frac{Q\, d\varphi}{2\pi}$ $\lambda = \frac{Q}{2\pi R}$ $dl = R\,d\varphi$ $\varphi \in [0, 2\pi)$

$$V = \frac{Q}{8\pi^2\epsilon_0}\int_0^{2\pi}\frac{d\varphi}{\sqrt{R^2+x^2}}$$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0\sqrt{R^2+x^2}}}$$

b) $\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$ since $\frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0$

$$\vec{E} = E_x\hat{x} = \frac{-Q}{4\pi\epsilon_0}\left[\frac{\partial}{\partial x}\frac{1}{\sqrt{R^2+x^2}}\right]\hat{x} = \frac{-Q}{4\pi\epsilon_0}\left[-\frac{1}{2}(R^2+x^2)^{-3/2}(2x)\right]\hat{x}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{x}{(R^2+x^2)^{3/2}}\hat{x}}$$ no y or z components of E

c) use part a, so $V_{disc} = \int_0^R dV_{ring}$

each ring has a different radius so $R \to r$, $r \in [0,R]$, $Q \to dQ$

$dV_{ring} = \frac{dQ}{4\pi\epsilon_0\sqrt{r^2+x^2}}$; $dQ = \sigma\, da = \sigma\, r\,dr\,d\varphi = a r^3\,dr\,d\varphi$

$$V_{disc} = \frac{a}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi}\frac{r^3\,d\varphi\,dr}{\sqrt{r^2+x^2}} = \frac{a}{2\epsilon_0}\int_0^R\frac{r^3\,dr}{\sqrt{r^2+x^2}}$$

$$= \frac{a}{2\epsilon_0}\left[\left(r^2\sqrt{r^2+x^2}\right)\Big|_0^R - \int_0^R 2r\sqrt{r^2+x^2}\,dr\right]$$

$$= \frac{a}{2\epsilon_0}\left[R^2\sqrt{R^2+x^2} - \frac{2}{3}(r^2+x^2)^{3/2}\Big|_0^R\right]$$

$$\boxed{V_{disc} = \frac{a}{2\epsilon_0}\left[R^2\sqrt{R^2+x^2} - \frac{2}{3}(R^2+x^2)^{3/2} + \frac{2}{3}x^3\right]}$$

by parts

$u = r^2$

$du = 2r\,dr$

$dv = \frac{r\,dr}{\sqrt{r^2+x^2}}$

$v = \sqrt{r^2+x^2}$

$\int u\,dv = uv - \int v\,du$

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Grading Scheme for Problem #3 and Comments

a) 3 pts correct $dq = \frac{Q}{2\pi} d\varphi$

2 " " $V = \int \frac{dq}{4\pi\epsilon_0 r}$ (1 pt if $V = -\int \vec{E} \cdot d\vec{l}$ + didn't get correct)

2 " " $r = \sqrt{R^2 + x^2}$

1 " " limits (depended on how you did it, just had to make sense + work. We weren't integrating over r or x, so ~~those~~ no pts for that).

2 " " integration

- only -1 for careless mistakes (being off by a factor, etc.)

☆ Gauss's law cannot be used here! It can only be applied to long cylinders, infinite planes, parallel plate capacitor set ups (where fringe fields are ignored), spheres and point charges. $\vec{E}$ is not uniform at P. Same goes for part 3c!!!!

- key is to realize that it is much easier to calculate $V = \int \frac{dq}{4\pi\epsilon_0 r}$ than $V = -\int \vec{E} \cdot d\vec{l}$, b/c $\vec{E}$ is harder to calculate from $\vec{E} = \int \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$ b/c it is a vector quantity, not a scalar like V.
- 1 pt for the correct answer with no work. Maybe you did it in your head. Maybe you copied it from a notecard. This exam (as are all physics tests) was about showing you know your stuff by showing work!

b) 2 pts $E = -\nabla V$

3 " $E_y = E_z = 0$

5 pts differentiation

OR

2 pt dq

1 $\vec{E} = \int \frac{dq}{4\pi\epsilon_0 r^2}\hat{r}$

2 " y, z cancel

2 pts r

1 limits

2 integration

- if part a was wrong, but you correctly differentiated, no pts were taken off.

c) 2 pts dQ

2 " r

2 " $V = \int \frac{dq}{4\pi\epsilon_0 r}$

1 " limit

3 integration

- Gauss's law only works for $x \ll R$, not any x
- once again $V = -\int \vec{E} \cdot d\vec{l}$ is not the way to go — you cannot blindly use formulas like this or $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$. It takes experience to recognize which situations warrant which techniques...