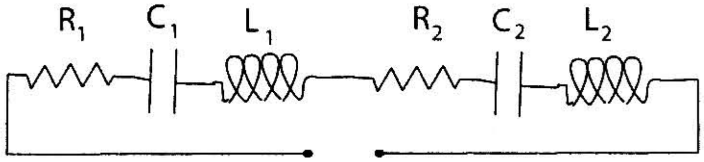
Packard - Fall '04

Lecture 3

Section 3

Prob 1 (both)

1. (10 points) A circuit has two resistors R_1 and R_2, two capacitors C_1 and C_2 and two inductors L_1 and L_2 all connected in series. Find the resonant frequency of the circuit.



$$W_{res} = \frac{1}{\sqrt{L_{eq} C_{eq}}} \quad (+5)$$

$$L_{eq} = L_1 + L_2 \quad (+2)$$

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \quad (+2)$$

$$W_{res} = \sqrt{\frac{C_1 + C_2}{(L_1 + L_2) C_1 C_2}} \quad (+1)$$

If the student used the frequency a DC LRC circuit would have: $\omega' = \sqrt{\omega_0^2 - \frac{L^2}{4R^2}}$ then they got no points (out of 5) for the W_{res} portion. If a math error was made usually 1 pt was deducted and if they used $C_{eq} = \frac{1}{C_1} + \frac{1}{C_2}$ they got 0/2 for C_{eq} but got 1/1 if they plugged their C_{eq} in correctly fir W_{res}.

Problem 2

Resistance ($\mathbb{R}$) of solenoid

N.B: N = total number of loops = $\frac{\ell}{d}$

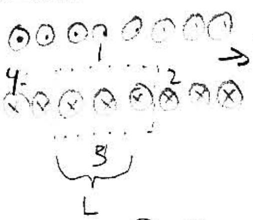
$$\mathbb{R} = \rho \times \frac{\text{Length}}{\text{Area}}$$

$$\text{Length} = N \times 2\pi R = \frac{\ell}{d} 2\pi R = 2\pi \frac{R\ell}{d}$$

$$\text{Area} = \pi \left(\frac{d}{2}\right)^2 = \frac{1}{4}\pi d^2$$

$$\Rightarrow \mathbb{R} = \rho \frac{2\pi \frac{R\ell}{d}}{\frac{1}{4}\pi d^2} = \frac{8R\ell\rho}{d^3}$$

B field of solenoid



Using the amperian loop shown we calculate the B field in the solenoid

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 \frac{N}{\ell} L I = \mu_0 \frac{1}{d} L I$$

By symmetry $\vec{B} \perp d\vec{\ell}$ on 2 and 4 so they contribute nothing to the integral. Furthermore B is weak outside the solenoid so $\int_3 \vec{B}\cdot d\vec{\ell} \approx 0$. Thus only segment 1 matters.

$$\Rightarrow \int_1 \vec{B}\cdot d\vec{\ell} = BL = \mu_0 \frac{1}{d} L I$$

$$\Rightarrow \vec{B} = \mu_0 \frac{1}{d} I \hat{z}$$

L of solenoid

using $\vec{B}$ we find

$$\Phi_B = N\,\Phi_{loop} = N\int_{loop}\vec{B}\cdot d\vec{a} = NB\pi R^2$$
$$= \frac{l}{d}\left(\mu_0 \frac{1}{d} I\right)\pi R^2$$
$$= \pi R^2 \mu_0 \frac{l}{d^2} I$$

$$L = \frac{|\mathcal{E}|}{\dot{I}} = \frac{\dot{\Phi}_B}{\dot{I}} = \boxed{\pi R^2 \mu_0 \frac{l}{d^2}}$$

I, B as fcn. of time

As we know, for an LR circuit

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \qquad \tau = \frac{L}{R}$$

This is how current will ramp up for this circuit since we can treat the inductance and the resistance of the solenoid in parallel. Thus we have...

$$B(t) = \frac{\mu_0}{d} I(t) = \frac{\mu_0}{d}\left(\frac{V_0}{R}\right)\left(1 - e^{-\frac{t}{\tau}}\right)$$
$$= \mu_0 V_0 \left(\frac{d^{3/2}}{8Rlp}\,\frac{1}{d}\right)\left(1 - e^{-\frac{t}{\tau}}\right)$$
$$= \frac{\mu_0 V_0 d^2}{8Rlp}\left(1 - e^{-t/\tau}\right)$$

$$\tau = \frac{L}{\mathbb{R}} = \frac{d^3}{8R\ell\rho} \times \frac{\mu_0 \pi R^{\cancel{2}}\cancel{\ell}}{\cancel{d^2}} = \frac{\mu_0 \pi d R}{8\rho}$$

<u>radius of particle (r)</u>

Centrifical force felt by partical in loop $= e u B$

N.B. the magnitude of the particle will remain constant b/c B-field does no work.

$$\Rightarrow \quad ma \overset{\text{circular motion}}{=} m\frac{u^2}{r} = euB \quad \text{N.B. } e = -|e| < 0$$

$$\Rightarrow r(t) = \frac{mu}{eB(t)} = \frac{mu}{e}\left(\frac{8R\ell\rho}{\mu_0 V_0 d^2}\right)\left(\frac{1}{1-e^{-t/\tau}}\right)$$

<u>The home stretch . . .</u>

Solve for t_0..

$$r(t_0) = \frac{R}{2}$$

$$\frac{R}{2} = \frac{8muR\ell\rho}{\mu_0 e V_0 d^2}\left(\frac{1}{1-e^{-t_0/\tau}}\right)$$

$$\Rightarrow 1 - e^{-t_0/\tau} = \frac{16mu\ell\rho}{\mu_0 e V_0 d^2}$$

$$\Rightarrow e^{-t_0/\tau} = 1 - \frac{16mu\ell\rho}{\mu_0 e V_0 d^2}$$

$$\Rightarrow -\frac{t_0}{\tau} = \ln\left(1 - \frac{16 m u l \rho}{\mu_0 e V_0 d^2}\right)$$

$$\Rightarrow t_0 = -\tau \ln\left(1 - \frac{16 m u l \rho}{\mu_0 e V_0 d^2}\right)$$

$$t_0 = -\frac{\mu_0 \pi d R}{8\rho} \ln\left(1 - \frac{16 m u l \rho}{\mu_0 e V_0 d^2}\right)$$

problem (#3)

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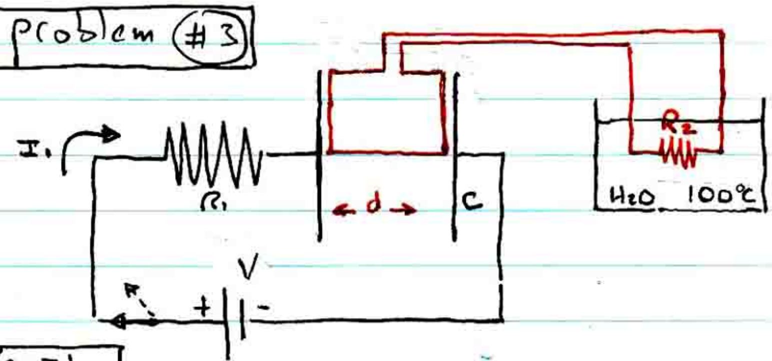


Key Ideas

5 (I) $\boxed{\text{charging Capacitor RC circuit solution}}$: $I_1(t) = \frac{V}{R_1} e^{-t/R_1C}$ (26-6 p 671)

(II) a/ $\boxed{\text{Ampere + displacement current}}$ $\oint \bar{B}\cdot d\bar{l} = \mu_0 I_{encl.} + \underbrace{\mu_0 \varepsilon_0 \frac{d\phi_E}{dt}}_{\text{displacement current } I_D(t)\text{encl.}}$

(5) → (III) $\boxed{\text{Short cut}}$: recall { displ. current $\boxed{I_D(t) = I_1(t)}$ ← as derived in lecture

[see next page for sol without shortcut] | displ current evenly distributed between cap // plates

$\rightarrow I_D(r,t) = I_1(t)\frac{r^2}{b^2}$

5 (II) b/ Back to $\boxed{\text{Ampere}}$ $\oint \bar{B}\cdot d\bar{l} = \mu_0 I_{D encl.}$

$B\, 2\pi r = \mu_0 I_1(t)\frac{r^2}{b^2}$

$\vec{B}(r,t) = \frac{\mu_0}{2\pi b^2} I_1(t)\, r\, \hat{\theta}$

$\rightarrow [B] = [\mu_0 I]/L$ OK

← B field induced between cap plates due to I_D – B field is circular, centered around 'axis' of cap and prop to r (dist from center)

5 (IV) $\boxed{\text{Faraday}}$: a time varying $\bar{B}$ field induces an Emf : $\mathcal{E}$

→ or a time varying B flux through loop induces an Emf around loop

B flux → $\phi_B = \int_0^b B(r,t)\, \underbrace{d\, dr}_{\text{area el.}} = \int_0^b \underbrace{\frac{\mu_0}{2\pi b^2} I_1(t)\, r}_{B(r,t)}\, \underbrace{d\, dr}_{\text{area el}}$ $[\phi_B] = [\mu_0 I]\,L$ OK

$\phi_B = \frac{\mu_0 d I_1(t)}{2\pi b^2} d\frac{r^2}{2}\Big|_0^b = \frac{\mu_0 I_1(t) d \cancel{b^2}}{2\pi \cancel{b^2}\, 2} = \frac{\mu_0 d}{4\pi} I_1(t) = \frac{\mu_0 d}{4\pi}\frac{V}{R_1} e^{-t/R_1C}$

ignore sign / we only need abs value of $\mathcal{E}$

$|\mathcal{E}| = \frac{d}{dt}\phi_B = \frac{\mu_0 d}{4\pi}\frac{V}{R_1}\frac{d}{dt}e^{-t/R_1C} = \frac{\mu_0 d}{4\pi}\frac{V}{R_1}\left(\frac{-1}{R_1C}\right)e^{-t/R_1C} = \frac{\mu_0 d V}{4\pi R_1^2 C} e^{-t/R_1C}$

$\rightarrow [\mathcal{E}] = [\mu_0 I]\frac{L}{s}$ OK

5 (V) $\boxed{\text{Ohm}}$ $\mathcal{E}$ in loop drive current I_2 through R_2: $I_2 = \mathcal{E}/R_2$

Power dissipated in $R_2 = P_2 = \mathcal{E}^2/R_2 = \frac{[(\mu_0 dV/4\pi R_1^2 C)]^2}{R_2}[e^{-t/R_1C}]^2$

Heat generated in R_2: $\Delta Q = \int_0^\infty P_2\, dt = \int_0^\infty \{\ \underbrace{\quad}_{\ast}\ \} e^{-2t/R_1C} dt$

$$\boxed{\Delta Q = \left\{\left(\frac{\mu_0 d V}{4\pi R_1^2 C}\right)^2 \frac{1}{R_2}\right\}\left(\frac{R_1 C}{-2}\right) e^{-2t/R_1C}\Big|_0^\infty = \frac{1}{2}CV^2\left(\frac{\mu_0 d}{4\pi R_1 C}\right)^2 \frac{1}{R_1 R_2}}$$

Thermo Stuff

5 (VI) $\boxed{\text{Latent heat}}$: Mass Evap = $\boxed{M_W = \frac{\Delta Q}{L} = \frac{1}{2}CV^2\left(\frac{\mu_0 d}{4\pi R_1 C}\right)^2 \frac{1}{R_1 R_2 L}}$

$\rightarrow [L] =$ Joules Kg^{-1}

(VII) $\boxed{\text{Entropy}}$: constant temp → no $\int$ → $\boxed{\Delta S = \frac{\Delta Q}{T = (100+273)°K} = \frac{\Delta Q}{373}}$

III' Without short cut :

Charging Cap induces an increasing E field → increasing E flux

Volt across Cap $V_c(t) = V\,(1 - e^{-t/R_1C})$ (26-5b p670)

E field between // plates $= \frac{V_c}{d}$

$$\phi_E(r,t) = \pi r^2 E(t) = \frac{\pi r^2}{d} V\,(1 - e^{-t/R_1C})$$

$$\frac{d\phi_E}{dt} = \frac{\pi r^2 V}{d}\left(-\frac{1}{R_1C}\right)\left(-e^{-t/R_1C}\right) = \frac{\pi r^2}{d}\frac{V}{R_1C}\,e^{-t/R_1C}$$

Faraday: $\underbrace{\oint \bar{B}\,d\bar{l}}_{B\,2\pi r\;=} = \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$ ✓

$$B(r,t) = \frac{1}{\cancel{2\pi r}}\,\mu_0 \varepsilon_0 \frac{\cancel{\pi r^2}}{d}\frac{V}{R_1C}\,e^{-t/R_1C}$$

$$B(r,t) = \frac{\mu_0 \varepsilon_0 V r}{2dR_1}\frac{e^{-t/R_1C}}{C}$$

Recall Cap // plate : $C = \varepsilon_0 \frac{A}{d} = \varepsilon_0 \frac{\pi b^2}{d}$ → sub

$$B(r,t) = \frac{\mu_0 \cancel{\varepsilon_0} V r}{2\cancel{d}R_1}\frac{e^{-t/R_1C}}{\cancel{\varepsilon_0}\pi b^2/\cancel{d}} = \frac{\mu_0 r}{2\pi b^2}\frac{V}{R_1}e^{-t/R_1C} =$$

$$\boxed{B(r,t) = \frac{\mu_0}{2\pi b^2} I_1(t)\, r}$$ ← as in step II b

VIII Check Units

$[M_W] = \text{kg}$ ← we need

$$\left[\underbrace{\tfrac{1}{2}CV^2}_{\substack{\text{Energy}\\ = \\ \text{Joules}}}\left(\frac{\mu_0 d}{4\pi \underbrace{R_1C}_{\substack{\text{Time}\\ \text{sec}}}}\right)^2 \frac{1}{R_1R_2}\right] = \cancel{J}\,[\mu_0]^2\frac{L^2}{S^2}\frac{1}{\Omega^2}\frac{\text{kg}}{\cancel{J}} = \underbrace{\left(\frac{[\mu_0]^2 L^2}{S^2\Omega^2}\right)}_{\text{need to show } = 1}\text{kg}$$

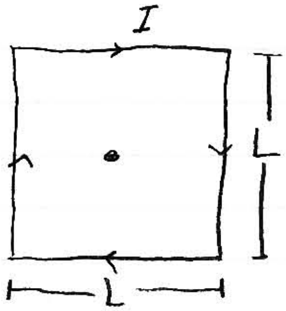
$[B] = \left[\frac{\mu_0 I}{r}\right] \to [\mu_0] = \left[\frac{Br}{I}\right] = \text{Tesla}\cdot\frac{L}{\text{Amp}}$

$\Omega = \frac{Volt}{A} = \frac{\text{Joules}}{A^2\,\text{sec}}$

$F = qvB$ — $[B] = \frac{N}{C\,L/s} = \frac{J}{AL^2} = T$

$$\underbrace{\frac{(T\,L/A)^2 L^2}{S^2[J/(A^2S)]^2}}_{=} = \frac{\cancel{J^2}}{\cancel{A^2L^4}}\frac{\cancel{L^2}}{\cancel{A^2}}\frac{\cancel{L^2}\cancel{A^4}\cancel{S^2}}{\cancel{S^2}\cancel{J^2}} = 1 \checkmark$$

4.

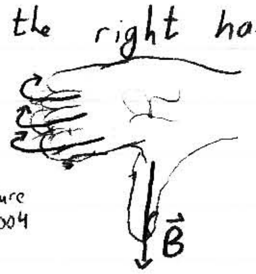


First, look at the symmetry of the problem:

We see that there will only be a B field into or out of the page @ the center.

Use the right hand rule to find the direction

$\Rightarrow$ B goes <u>into</u> the page.

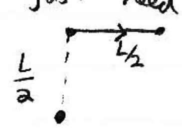


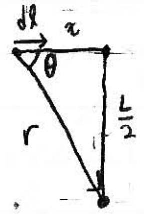
Picture © 2004

Instead of going over the entire wire, we can just use symmetry again.

The z-components of each side will be the same. Additionally, The z-compoent of one half of one side will be the same as the other half.

$\therefore$ we just need to find the z-component from:





$$d\vec{B} = \frac{\mu_0 I \, d\vec{l} \times \vec{r}}{4\pi r^3}$$

$$r = \sqrt{\frac{L^2}{4} + x^2} = \frac{1}{2}\sqrt{L^2 + 4x^2}$$

$$dl = dx$$

$$d\vec{l} \times \vec{r} = -dl \, r \sin\theta \, \hat{z}$$

$$= \cancel{r} \cdot \frac{L/2}{\cancel{r}} \, dl \, (-\hat{z})$$

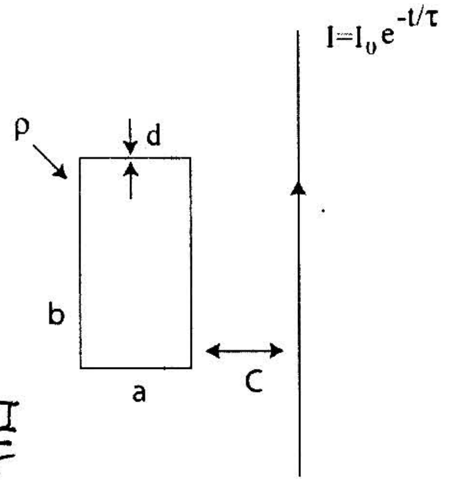
$$d\vec{B} = \frac{-\mu_0 I L \hat{z}}{2 \cdot 4\pi} \frac{dl}{r^3}$$

$$\Rightarrow \vec{B} = 8 \cdot \frac{-\mu_0 I L \hat{z}}{8\pi} \int_{-\frac{L}{2}}^{0} \frac{dx}{\frac{1}{8}(L^2 + 4x^2)^{3/2}} = -\frac{8 \mu_0 I L}{\pi} \int_0^{\frac{L}{2}} \frac{dx}{(L^2 + 4x^2)^{3/2}} \hat{z}$$

$$= \frac{-8\mu_0 I \cancel{L}}{\pi} \frac{\sqrt{2}}{4L^{\cancel{2}}} \hat{z} = \frac{-2\sqrt{2}\,\mu_0 I}{\pi L}$$

Section 3
Prob 5

5. (20 points) Wire with resistivity ρ and diameter d is bent into a rectangular loop with sides a and b as shown below. The loop of wire lays in a plane a distance c from a very long straight wire carrying a current of $I_o e^{-t/\tau}$. Compute both the net force and the net torque on the loop at time t=τ. For the direction of current shown, include the direction of the net force.



- changing current creates changing magnetic flux through loop
- EMF is induced
- induced current flows
- current in external magnetic field gives force

$B = \frac{\mu_o I}{2\pi r}$

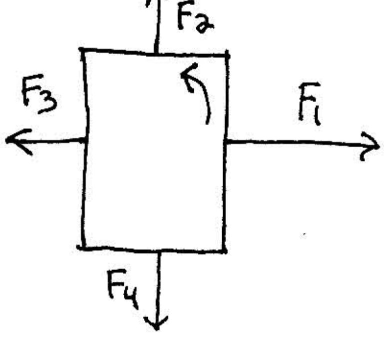
$$\Phi_B = \int \vec{B}\cdot d\vec{A} = \int_c^{c+a} \left(\frac{\mu_o I}{2\pi r}\right)(b\,dr) = \frac{\mu_o b I}{2\pi} \ln\left(\frac{c+a}{c}\right)$$

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\frac{I_o \mu_o b e^{-t/\tau}}{2\pi} \ln\left(\frac{c+a}{c}\right)\right) = \frac{\mu_o b}{2\pi} \ln\left(\frac{c+a}{c}\right) \frac{e^{-t/\tau}}{\tau} I_o$$

when $t = \tau$ $\boxed{\mathcal{E} = \frac{\mu_o b}{2\pi} \ln\left(\frac{c+a}{c}\right) \frac{I_o}{e\tau}}$

$I_{ind} = \mathcal{E}/R$ $\quad R = \frac{\rho L}{A} = \frac{\rho\, 2(a+b)}{\pi (d/2)^2} = \frac{8\rho(a+b)}{\pi d^2}$ $\quad \boxed{R = \frac{8\rho(a+b)}{\pi d^2}}$

Current is ↓ so flux out of page ↓ so we want to create a flux out of page ⟹ counterclockwise current



$F_2 = F_4$ they cancel

$F_1 = I_{ind}\, b\, B(r=c)$ $\quad F_3 = I_{ind}\, b\, B(r=c+a)$

$\Sigma F = F_1 - F_3 = I_{ind}\, b\, (B(r=c) - B(r=c+a))$

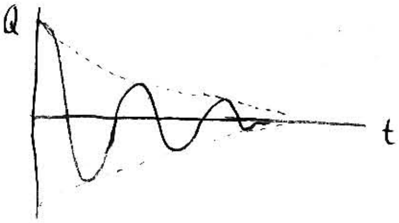
$\boxed{\Sigma F = \frac{\mathcal{E} b}{R} \frac{\mu_o I_o}{2\pi e}\left(\frac{1}{c} - \frac{1}{c+a}\right)}$ $\quad \mathcal{E}, R$ defined above

All forces in plane of page so $\boxed{\Sigma \tau = 0}$

Problem 6

Section 3
Prob 6
(both)

a) $\boxed{12 \text{ pts}}$ Charge should decay like for under damped case.



The Energy in the circuit can be given by

$$\frac{Q^2}{2C} + \frac{1}{2} L i^2 = \text{Total energy}$$

The equation for the charge decay is, in general:

$$\underline{4\text{pts}} \quad Q(t) = Q_0 e^{-\frac{R}{2L}t} \cos(\omega' t + \phi) \qquad \text{with} \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \underline{\text{pt } 1}$$

(for use in b)

This can come from cheat sheet or from the solution of the Diff eqn from applying KVL to the circuit

$$-L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

And the solution is the above equation

We can approximate Q from the decaying exponential alone.

1pt We know (since one can take the derivative) that the current also decays as $e^{-\frac{R}{2L}t}$

This means that we can see the energy in the circuit as the decaying exponential Q in the expression

$$\frac{Q^2}{2C} = \text{Energy}$$

The idea here is that the envelope reflects the total energy of the capacitor and inductor added together as an effective charge.

Thus $$\frac{\left(Q_0 e^{-\frac{R}{2L}t}\right)^2}{2C} = \text{Energy}(t)$$

2 pts

So, the half energy is $\frac{Q_0^2}{4C}$ 2 pts

Thus, $$\frac{\cancel{Q_0^2} e^{-\frac{R}{L}t}}{\cancel{2C}} = \frac{\cancel{Q_0^2}}{2\cancel{4C}} \longrightarrow e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$-\frac{R}{L}t = \ln\left(\tfrac{1}{2}\right)$$

$$\boxed{t = \frac{L}{R}\ln 2} \quad \underline{2\text{ pts}}$$

b) $\boxed{3 \text{ pts}}$ $\omega_1 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ from above

$= 1.053 \times 10^3$ rad/s $\underline{1\text{pt}}$

$$\omega = \frac{1}{\sqrt{LC}} = 1.118 \times 10^3 \qquad \frac{\omega_1}{\omega} = .942 \quad \underline{1\text{pt}}$$

$\underline{1\text{pt}}$