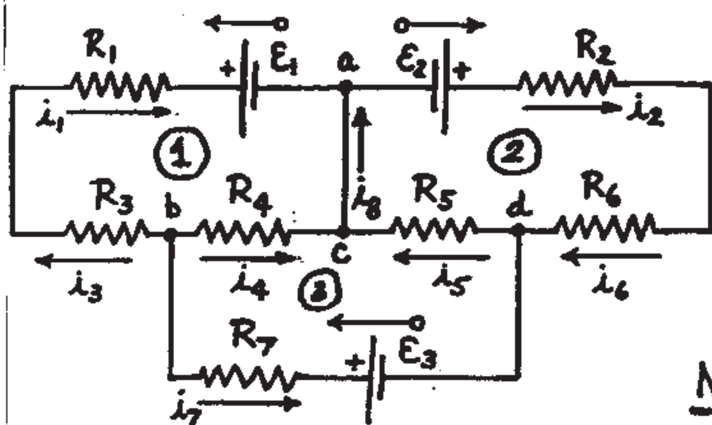
Physics 7B (Sec. 2) Solutions to Midterm #2 April 8, 2003

(1) Number the circuit loops as shown and apply Kirchoff's Loop Theorem, traversing each loop in a clockwise direction, resulting in three equations:



(1) $-i_1R_1 - \mathcal{E}_1 + i_4R_4 - i_3R_3 = 0$
(2) $\mathcal{E}_2 - i_2R_2 - i_6R_6 - i_5R_5 = 0$
(3) $-i_4R_4 + i_5R_5 + \mathcal{E}_3 + i_7R_7 = 0$

Next, apply Kirchoff's Junction Theorem to junctions, a, b, c, d, obtaining four equations:

(4) $i_1 + i_8 - i_2 = 0$; (5) $-i_3 - i_7 - i_4 = 0$;

(6) $i_4 - i_8 + i_5 = 0$; (7) $-i_5 + i_6 + i_7 = 0$

Equations (1)-(7) are seven equations in eight unknowns, so at least one additional equation is required for solution. A further equation may be obtained from (8)

$$i_3 = i_1 \quad \text{or} \quad i_2 = i_6 .$$

Note that i_8 does not appear in the equations for loops 1 and 2 because i_8 does not pass through a resistor. Thus i_8 does not contribute a potential difference to the Kirchoff loop equations for loops 1 and 2.

(2) (a) The capacitance C of parallel plates separated by air is

$$C = (\epsilon_0 A/d)$$

where the area A of the plates is $A = \pi r^2 = \pi(0.082)^2 = 2.11\times10^{-2}\ m^2$ and the separation $d = 1.3\times10^{-3}$ meters; $\epsilon_0 = 8.85\times10^{-12}\ F\,m^{-1}$, so

$$C = 1.44\times10^{-10}\ \text{Farad}$$

The charge Q on either plate is given by $Q = CV$, where the potential difference V between the plates is 120 volts. Then

$$Q = CV = (1.44\times10^{-10})(120) = 1.73\times10^{-8}\ \text{Coulombs} = 17.3\ nC$$

(b) With a dielectric of dielectric constant K inserted, the capacitance C′ is given by

$$C' = KC = (11.7)(1.44\times10^{-10}) = 16.8\times10^{-10}\ \text{Farads}$$

The charge on either plate is still Q because (with battery B removed) there is no conducting path (by which charge might move) between the plates, so

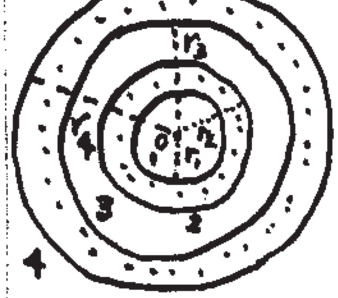
$$Q = C'V' \Rightarrow V' = (Q/C') = \frac{1.73\times10^{-8}}{16.8\times10^{-10}} = 10.3\ \text{volts}$$

(c) With battery B replaced (now the capacitor has capacitance C′) charge can flow (through the battery) between the plates, so

$$Q' = C'V = (16.8\times10^{-10})(120) = 2.02\times10^{-7}\ C. = 202\ nC$$

(d) $Q' \neq Q$ because, when the battery B is replaced, charge flows from one plate of the capacitor to the other until (after a long time) the final charge Q′ accumulates on either plate.

(3)(a)



Since the shells are <u>conducting</u>, the electric field $|\underline{E}|=0$ inside the shells and all excess charge resides on the <u>surfaces</u> of the shells. An excess charge $(+Q)$ is placed on the inner shell.

<u>To find the charge Q_1 on surface 1</u> (the inner surface of the inner shell), construct a spherical Gaussian surface (concentric with the shells) of radius R, where $r_2 > R > r_1$. This Gaussian surface is inside the inner shell. Since the electric field E_1 inside the inner shell is $E_1 = 0$, Gauss' Law becomes

$$\epsilon_0 \oint \underline{E}_1 \cdot d\underline{A} = Q_1$$

leading to

$$\boxed{Q_1 = 0.}$$

<u>To find the charge Q_2 on surface 2</u>, (the outer surface of the inner shell), note that the problem tells us that there is a charge $(+Q)$ on the inner shell. Since the shell is conducting, all of this charge must be on the outer surface 2, so

$$\boxed{Q_2 = +Q.}$$

<u>To find the charge Q_3 on surface 3</u> (the inner surface of the outer shell), construct a Gaussian sphere of radius R', where $r_3 < R' < r_4$, so this Gaussian sphere is inside the outer shell. The electric field E_2 inside the outer shell is $E_2 = 0$, so Gauss' Law becomes

$$\epsilon_0 \oint E_2 \cdot d\underline{A} = Q'$$

(continued →)

(3)(a)[continued] where Q' is the total charge enclosed by this Gaussian surface. Therefore

$$Q' = Q_1 + Q_2 + Q_3 .$$

Since $E_2 = 0$, Gauss' Law gives us $Q' = 0$, so, with $Q_1 = 0$, $Q_2 = +Q$,

$$0 = 0 + Q + Q_3$$

or

$$\boxed{Q_3 = -Q}$$

for the charge on surface ③.

To find the charge Q_4 on surface ④ (the outer surface of the outer shell), we note that the net excess charge Q'' on the outer shell is zero, because excess charge was placed only on the inner shell. Since

$$Q'' = Q_3 + Q_4 = 0$$

and

$$0 = -Q + Q_4$$

$$\boxed{Q_4 = +Q}$$

is the charge on the outer surface ④ of the outer shell.

As a check, note that the total excess charge on the entire system is

$$Q_1 + Q_2 + Q_3 + Q_4 = Q$$

which is the value it should be.

(3)[continued] (b) We want to calculate the potential difference $(V_3 - V_2)$ between surfaces 3 and 2, so we need the electric field $\underline{E}$ in the space (vacuum) between surfaces 3 and 2. Since the conductors are spherical and $\underline{E}$ is normal to the conducting surfaces, $\underline{E}$ is radial in direction. We use Gauss' Law to calculate $\underline{E}$, drawing a spherical Gaussian surface of radius r, where $r_3 > r > r_2$. Gauss' Law is then

$$\epsilon_0 \int \underline{E} \cdot dA = \epsilon_0 E(r) \int dA = \epsilon_0 E(r)[4\pi r^2] = (Q_1 + Q_2) = Q$$

since $(Q_1 + Q_2) = Q$ is the net charge enclosed by the Gaussian surface. Therefore

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}.$$

The potential difference $(V_3 - V_2)$ is given by

$$(V_3 - V_2) = -\int \underline{E} \cdot d\underline{l}$$

taken over $r_3 > r > r_2$. Here $d\underline{l} = (dr)\hat{u}_r$ and the radial $\underline{E}(r) = E(r)\hat{u}_r$, so $(\underline{E} \cdot d\underline{l}) = E(r)dr$ because $(\hat{u}_r \cdot \hat{u}_r) = 1$. Then the integral is

$$(V_3 - V_2) = -\int_{r_2}^{r_3} E(r)dr = -\int_{r_2}^{r_3} \frac{Q}{4\pi\epsilon_0 r^2} = -\frac{Q}{4\pi\epsilon_0}\int_{r_2}^{r_3} \frac{dr}{r^2} = \frac{-Q}{4\pi\epsilon_0}\left|-\frac{1}{r}\right|_{r_2}^{r_3}$$

leading to

$$\boxed{(V_3 - V_2) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_3} - \frac{1}{r_2}\right]}$$

(c) Since $r_3 > r_2$, $(1/r_3) < (1/r_2)$; since $Q > 0$, $(V_3 - V_2) < 0$ so

$$\boxed{V_2 > V_3}$$

and surface 2 is at a higher potential than surface 3. This agrees with $Q_2 = +Q$ and $Q_3 = (-Q)$.

(continued →)

(3)[continued] (d) To calculate the capacitance C of the system composed of surfaces (2) and (3), we use the definition of capacitance

$$C = (Q/V)$$

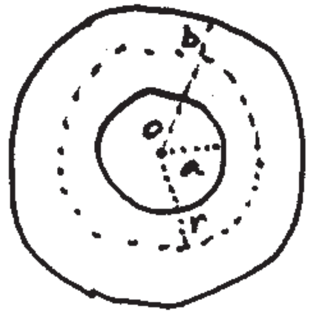
which in this case is

$$C = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left|\left(\frac{1}{r_3} - \frac{1}{r_2}\right)\right|}$$

$$\boxed{C = \frac{4\pi\epsilon_0}{\left|\left(\frac{1}{r_3} - \frac{1}{r_2}\right)\right|}}$$

as the required capacitance. (We use absolute value of $[(1/r_3) - (1/r_2)]$ because capacitance is always a positive number.

(4)(a) In order to calculate the energy density $u_E(r)$, we need to know the electric field magnitude $|\underline{E}(r)| \equiv E(r)$, which can be calculated using Gauss' Law. Take as the Gaussian surface a sphere of radius r, centered at O. Then Gauss' Law is



$$\epsilon_0 \oiint \underline{E} \cdot \hat{n}\, dA = q_{enc}$$

where q_{enc} is the charge enclosed within the Gaussian surface. To calculate q_{enc},

$$q_{enc} = \int \rho(r)\, dV = \int_a^r \rho_0 r^{-1} [4\pi r^2]\, dr \quad (\text{where } a \lesssim r);\ \rho(r) = \rho_0 r^{-1}$$

where the element of volume dV is a spherical shell of radius r, thickness (dr) and volume $dV = (4\pi r^2)\, dr$. Then

$$q_{enc} = \int_a^r 4\pi\rho_0 r\, dr = 4\pi\rho_0 \left[\frac{r^2}{2}\right]_a^r = 4\pi\rho_0\left(\frac{r^2}{2} - \frac{a^2}{2}\right) = 2\pi\rho_0 (r^2 - a^2)$$

Now that we have q_{enc}, we can use Gauss' Law to calculate $E(r)$. Since $\underline{E}(r)$ is radially outward in direction, $\underline{E}(r) = E(r)\,\hat{u}_r$. Since $E(r)$ does not depend on direction, we can write, $\hat{n} = \hat{u}_r$,

$$\epsilon_0 \oiint E(r)\,\hat{u}_r \cdot \hat{u}_r\, dA = \epsilon_0 \oiint E(r)\, dA = \epsilon_0 E(r) \oiint dA = q_{enc}$$

so, with $\oiint dA = 4\pi r^2$, we obtain

$$\epsilon_0 E(r) [4\pi r^2] = 2\pi\rho_0 (r^2 - a^2)$$

so $\boxed{E(r) = (\rho_0 / 2\epsilon_0)\left(1 - \frac{a^2}{r^2}\right)}$ with which we can calculate the energy density $u_E(r)$

(continued →)

(4)(a) [continued] The energy density $u_E(r)$ is then given by

$$u_E(r) = \tfrac{1}{2}\epsilon_0 \left|\underline{E}(r)\right|^2 = \tfrac{1}{2}\epsilon_0 \left(\rho_0^2/4\epsilon_0^2\right)\left(1-\frac{a^2}{r^2}\right)^2$$

$$\boxed{u_E(r) = \left(\rho_0^2/8\epsilon_0\right)\left(1-\frac{a^2}{r^2}\right)^2}$$

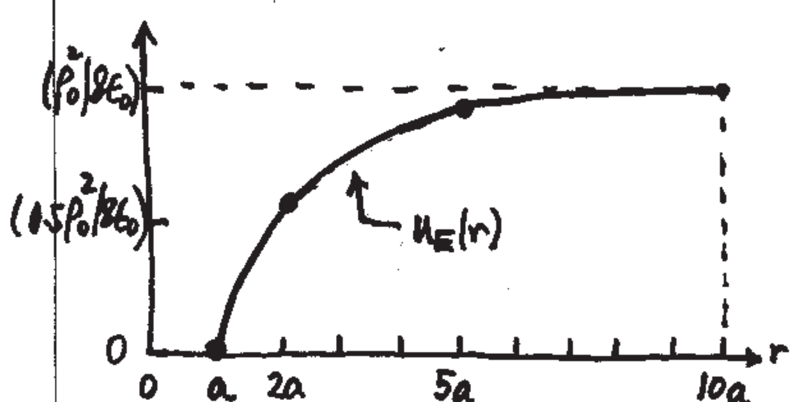
(b) For $b = 10a$, $(b/a) = 10$, $(b^2/a^2) = 100$, $(a^2/b^2) = 0.010$

Then the <u>minimum</u> value $u_E(\text{min})$ occurs for $r = a$: $\boxed{u_E(\text{min}) = 0}$

The <u>maximum</u> value $u_E(\text{max})$ occurs for $r = b$:

$$u_E(\text{max}) = \left(\rho_0^2/8\epsilon_0\right)\left(1-\frac{a^2}{b^2}\right)^2 = \left(\rho_0^2/8\epsilon_0\right)\left(1.00-0.01\right)^2 \cong \left(\rho_0^2/8\epsilon_0\right)$$

<u>Plot of $u_E(r)$ as a function of r</u> (for $b = 10a$):

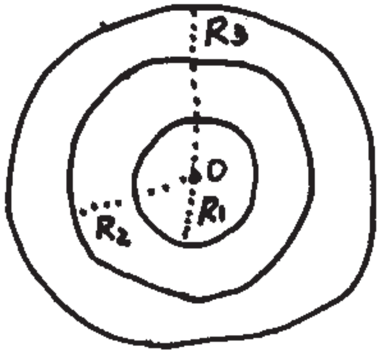


<u>For $r = 2a$</u>: $u_E(2a) = \left(\rho_0^2/8\epsilon_0\right)\left(1-\frac{a^2}{4a^2}\right)^2 = \left(\frac{\rho_0^2}{8\epsilon_0}\right)\left(1-\frac{1}{4}\right)^2 = (0.56)\frac{\rho_0^2}{8\epsilon_0}$

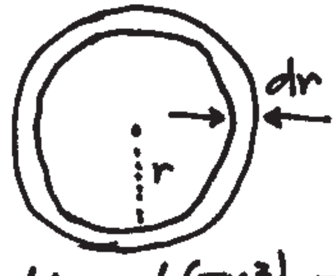
<u>For $r = 5a$</u>: $u_E(5a) = \left(\rho_0^2/8\epsilon_0\right)\left(1-\frac{a^2}{25a^2}\right)^2 = \left(\frac{\rho_0^2}{8\epsilon_0}\right)\left(1-\frac{1}{25}\right)^2 = (0.92)\frac{\rho_0^2}{8\epsilon_0}$

<u>Units Check</u>: $\rho_0^2 = C^2 m^{-4}$, $\epsilon_0 = C^2 N^{-1} m^{-2}$, $\boxed{(\rho_0^2/\epsilon_0) = N m^{-2} = (N\text{-}m) m^{-3} = J m^{-3}}$

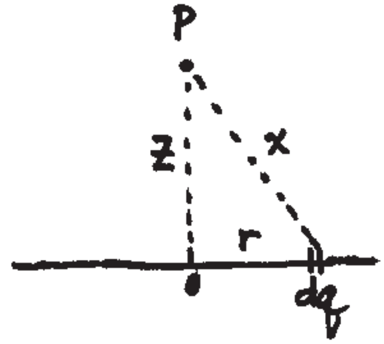
(5) Only the disc and the ring matter because the supporting strips are of negligible size. Take as element of area dA an annular ring of radius r and width dr:



TOP VIEW
(Point P is a perpendicular distance z above paper)



$$dA = d(\pi r^2) = 2\pi r\, dr$$



SIDE VIEW
(gaps between disc and ring are not shown)

$$x^2 = z^2 + r^2$$

Take as the element of charge dq (on ring and disc) the charge on the annular ring:

$$dq = \sigma\, dA = 2\pi \sigma r\, dr$$

The element dV of electric potential at point P due to element dq of charge at a distance x from dq is (treating dq as a point charge) given by

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{x} = \frac{2\pi\sigma r\, dr}{4\pi\epsilon_0 (z^2+r^2)^{1/2}} \quad \text{where} \begin{cases} 0 \le r \le R_1 \\ R_2 \le r \le R_3 \end{cases}$$

Since, for a given value of r, all points on an annular ring of radius r are at same distance x from point P, and since, for a given point P, z is fixed, the total electric potential V at point P is given by (since $dV \to 0$ as $z \to \infty$) (continued ⟶)

(5) [continued]

$$V(z) = \int_{\text{disc}} dV + \int_{\text{ring}} dV$$

so

$$V(z) = \int_0^{R_1} \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma\, r\, dr}{(z^2+r^2)^{1/2}} + \int_{R_2}^{R_3} \frac{1}{4\pi\epsilon_0} \frac{2\pi\sigma\, r\, dr}{(z^2+r^2)^{1/2}}$$

$$V(z) = \frac{2\pi\sigma}{4\pi\epsilon_0}\left[\int_0^{R_1} \frac{r\, dr}{(z^2+r^2)^{1/2}} + \int_{R_2}^{R_3} \frac{r\, dr}{(z^2+r^2)^{1/2}}\right]$$

Using one of the integrals supplied,

$$V(z) = \frac{\sigma}{2\epsilon_0}\left\{\left[(r^2+z^2)^{1/2}\right]_0^{R_1} + \left[(r^2+z^2)^{1/2}\right]_{R_2}^{R_3}\right\}$$

$$\boxed{V(z) = \frac{\sigma}{2\epsilon_0}\left[(R_1^2+z^2)^{1/2} - z + (R_3^2+z^2)^{1/2} - (R_2^2+z^2)^{1/2}\right]}$$

potential at point P; depends on z

(b) The component E_z of the electric field at P in the z-direction (normal to plane of ring and disc) is

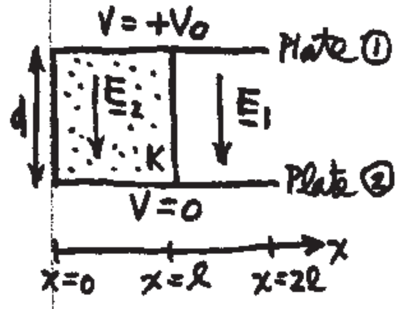
$$E_z = (-dV/dz)$$

so

$$\boxed{E_z = (-\sigma/2\epsilon_0)\left\{\frac{z}{(R_1^2+z^2)^{1/2}} - 1 + \frac{z}{(R_3^2+z^2)^{1/2}} - \frac{z}{(R_2^2+z^2)^{1/2}}\right\}}$$

(c) The negative sign above means that the electric field vector $\underline{E}_z$ at point P points in the direction opposite to the direction in which the potential $V(z)$ increases. Since $V(z)$ decreases with increasing z, so $\underline{E}_z$ points in the direction of increasing z.

(6)(a) The uniform electric field $\underline{E}_2$ in the dielectric <u>does</u> have the same magnitude as the uniform electric field $\underline{E}_1$ in the unfilled region of the capacitor. This can be proved as follows. Since the metal plates are equipotential surfaces, the potential difference between the plates is V_0 volts in both regions of the capacitor. From the definition of potential difference,



$$V_0 = -\int \underline{E} \cdot d\underline{l} = -E_2 d = -E_1 d$$

since the potential difference V_0 is the same in both regions of the capacitor. Thus

$$\boxed{E_1 = E_2}$$

(b) We know that the electric field $E_1 = (\sigma_1/\epsilon_0)$ where σ_1 is the surface charge density on the plates in the <u>unfilled</u> region ($l \le x \le 2l$) of the capacitor. Similarly, $E_2 = (\sigma_2/K\epsilon_0)$ for the <u>filled</u> ($0 \le x \le l$) region. Here K is the dielectric constant of the dielectric in the filled region and σ_2 is the surface charge density on the plates in the filled region. Since $E_1 = E_2$, we have

$$(\sigma_1/\epsilon_0) = (\sigma_2/K\epsilon_0) \implies \boxed{\sigma_2 = K\sigma_1}$$

so σ_2 is K times σ_1. <u>For the case</u> $K = 2$, $\boxed{\sigma_2 = 2\sigma_1}$ and the required plot is shown at left. The surface charge density σ_2 on the plates in the filled region is K ($=2$) times the surface charge density σ_1 on the plates in the unfilled region. Plate ② has the same surface charge densities of opposite sign.