Physics 7B <u>Solutions to First Exam</u> July 20, 1992

(1) Consider first the isothermal expansion from $p_i = 2.047 \times 10^5$ Nm^{-2} to $p_f = 1.013 \times 10^5\ Nm^{-2}$. Since $(p_i/p_f) = 2.02$, the final volume $V_f = 2.02 V_i = 0.283\ m^3$. The work done <u>by</u> the gas in this step is, using $pV = nRT$ for an ideal gas,

$$W_1 = nRT \ln(V_f/V_i) = pV \ln(V_f/V_i) = p_i V_i \ln(V_f/V_i)$$

$$W_1 = (2.047 \times 10^5)(0.14) \ln(2.02) = 2.02 \times 10^4 \text{ Joules}$$

Consider next cooling the gas at constant $p_f = 1.013 \times 10^5\ Nm^{-2}$ from $V_f = 0.283\ m^3$ to $V_i = 0.14\ m^3$. The work W_2 done by the gas is

$$W_2 = p_f(0.14 - 0.283) = (1.013 \times 10^5)(-0.143) = -1.45 \times 10^4 \text{ Joules}$$

Total work done <u>by</u> gas = $\boxed{W = W_1 + W_2 = 5700\ J.}$

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(2) <u>Process is</u>: $(1\ kg\ ice, 273 K) \longrightarrow (1\ kg\ water, 273 K)$: <u>irreversible</u>
To calculate ΔS_{irrev} for this irreversible process, we can calculate ΔS_{rev} for a <u>reversible</u> process connecting the same initial and final states because the entropy S of the system is a state variable. For a reversible process,

$$dS_{rev} = \frac{dQ}{T} \implies \Delta S_{rev} = \int \frac{dQ}{T} = \frac{1}{T}\int dQ = \frac{Q}{T}$$

because T = constant = 273 K. The heat $Q = 3.332 \times 10^5\ J$ for 1 kg of ice melting, so

$$\Delta S_{irrev} = \Delta S_{rev} = \frac{3.332 \times 10^5}{273} = 1220\ J(K)^{-1} \quad \boxed{\Delta S_{irrev} = 1220 \frac{J}{K}}$$

(3) We know that 0.1 kg of ice melts, so total heat Q involved in cyclic process is 3.332×10^4 Joules. For the three steps:

(1) Gas compressed quickly: $\Delta U_1 = Q_1 - W_1$
(2) Heat enters at const. volume: $\Delta U_2 = Q_2 - W_2$
(3) Gas expands slowly: $\Delta U_3 = Q_3 - W_3$

Since process is cyclic, $\Delta U = \Delta U_1 + \Delta U_2 + \Delta U_3 = 0$

so $(Q_1+Q_2+Q_3) - (W_1+W_2+W_3) = 0 \implies (W_1+W_2+W_3) = (Q_1+Q_2+Q_3)$

Since heat must leave gas to melt ice, $(Q_1+Q_2+Q_3) = -3.332 \times 10^4$ J.

$\therefore$ $(W_1+W_2+W_3)$ = work done by gas $= -3.332 \times 10^4$ Joules

Work done on gas $= -(W_1+W_2+W_3) = +3.332 \times 10^4$ Joules

(Energy leaves gas as heat and energy enters gas as work, so $\Delta U = 0$)

(4) Given $f(v)\,dv = 4\pi N (m/2\pi k_B T)^{3/2} v^2 e^{-mv^2/2k_BT}\,dv$

$$\overline{v^2} = \frac{1}{N}\int_0^\infty v^2 f(v)\,dv = 4\pi (m/2\pi k_B T)^{3/2} \int_0^\infty v^4 e^{-mv^2/2k_BT}\,dv$$

Evaluate integral using $\int_0^\infty x^4 e^{-r^2x^2}\,dx = (3\pi^{1/2}/8r^5)$

with $r = (m/2k_BT)^{1/2}$, giving $\int_0^\infty v^4 e^{-mv^2/2k_BT}\,dv = \left(\frac{3\pi^{1/2}}{8}\right)\left(\frac{2k_BT}{m}\right)^{\frac{5}{2}}$

Then $\overline{v^2} = 4\pi \left(\frac{m}{2k_BT\pi}\right)^{\frac{3}{2}} \left(\frac{3\pi^{1/2}}{8}\right)\left(\frac{2k_BT}{m}\right)^{5/2} = \frac{24 k_BT}{8m}$

$$\boxed{\overline{v^2} = (3k_BT/m)}$$

(5) The situation:

(a)

Electric field at P due to (+Q) is $\underline{E}_+ = \frac{1}{4\pi\epsilon_0}\frac{Q}{(r-a)^2}\hat{x}$

Electric field at P due to (−Q) is $\underline{E}_- = \frac{1}{4\pi\epsilon_0}\frac{(-Q)}{(r+a)^2}\hat{x}$

Total electric field $\underline{E}(r)$ at point P is $\underline{E}(r) = \underline{E}_+ + \underline{E}_-$, so

$$\boxed{\underline{E}(r) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right]\hat{x}}$$

(b) To expand $E(r)$, rewrite as $E(r) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r^2(1-\frac{a}{r})^2} - \frac{1}{r^2(1+\frac{a}{r})^2}\right]$

and expand $(1-\frac{a}{r})^{-2}$ and $(1+\frac{a}{r})^{-2}$ in Taylor Series. Then

$$(1-x)^{-2} \cong 1 + 2x + \dots \implies (1-\tfrac{a}{r})^{-2} \cong 1 + \tfrac{2a}{r}$$
$$(1+x)^{-2} \cong 1 - 2x + \dots \implies (1+\tfrac{a}{r})^{-2} \cong 1 - \tfrac{2a}{r}$$

with these expansions being valid when $x = (a/r) \ll 1$, or $r \gg a$.

Then

$$E(r) \cong \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r^2}\right]\left(1 + \frac{2a}{r} - \left[1 - \frac{2a}{r}\right]\right) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\left(\frac{4a}{r}\right)$$

Since the dipole moment $p = 2aQ$, we have

$$\boxed{|\underline{E}(r)| = E(r) \cong \frac{2p}{4\pi\epsilon_0}\frac{1}{r^3}}$$

so the magnitude of $\underline{E}(r)$ at a point P on axis of dipole at a distance $r \gg a$ from center of dipole axis. (Direction of $\underline{E}$ is along dipole (x) axis, parallel to $\hat{x}$.