Physics 57B — Solutions to Third Exam — Aug. 20, 1992

(1)(a) The charge $Q(t)$ is given in general by (G 32-14),

$$Q(t) = Q_0 \cos(\omega t + \varphi)$$

where Q_0 is the maximum charge on the capacitor. To determine Q_0, note that $Q(0) = 0$, so

$$0 = Q_0 \cos\varphi$$

leading to $\quad 0 = \cos\varphi \Rightarrow \varphi = -\frac{\pi}{2}$

(Note that $Q_0 \neq 0$ because $Q(0)=0$.) Hence we have

$$Q(t) = Q_0 \cos(\omega t - \tfrac{\pi}{2})$$

or

$$Q(t) = Q_0 \sin \omega t$$

because $\cos(\omega t - \frac{\pi}{2}) = \sin \omega t$. To determine Q_0, we are given

$$\left(\frac{dQ}{dt}\right)_{t=0} = I(0) = \omega Q_0 \cos 0 = \omega Q_0 = 2 \text{ Amps}$$

From $L = 3\times10^{-3}$ H, $C = 2.7\times10^{-6}$ F, $\omega = (LC)^{-1/2} = 1.11\times10^{4}\ \text{sec}^{-1}$

$$\Rightarrow \boxed{Q_0 = \frac{2}{\omega} = 1.8\times10^{-4} \text{ Coulomb}}$$

(b) We want the maximum value of (dU_E/dt), where $U_E = (Q^2/2C)$, so

$$U_E(t) = (1/2C)Q_0^2 \sin^2\omega t \Rightarrow (dU_E/dt) = (Q_0^2/2C)\frac{d}{dt}(\sin^2 \omega t)$$

$$(dU_E/dt) = (Q_0^2/2C)(2\omega)\sin\omega t \cos\omega t = (\omega Q_0^2/2C)\sin(2\omega t)$$

(continued →)

(2)(b) [continued] When $X_C = X_L$, $Z = R$, and $\langle P(t) \rangle = \max$.

so $\langle P(t) \rangle_{max} = \dfrac{\mathcal{E}_0^2 R}{2R^2} = \dfrac{\mathcal{E}_0^2}{2R} = \dfrac{(300)^2}{2(5)} = 9000$ Watts

(c) When $X_L = X_C$, $Z = R$, $\cos\varphi = (R/Z) = 1$, $\boxed{\varphi = 0}$

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(3) Coil forms a magnetic dipole of moment $\underline{\mu} = NI\underline{A} = NIA\hat{n}$

$\underline{\mu} = NIA\hat{n}$

$$\underline{\tau} = \underline{\mu} \times \underline{B}$$

$$\tau = \mu B \sin 90^\circ = NIAB$$

Angle between $\hat{n}$ and $\underline{B} = 90^\circ$ $\boxed{\tau = NIAB}$

Since length l of wire is fixed, area A depends on number N of turns in coil. For N loops, circumference of each loop is

$$c = (l/N) = 2\pi r$$

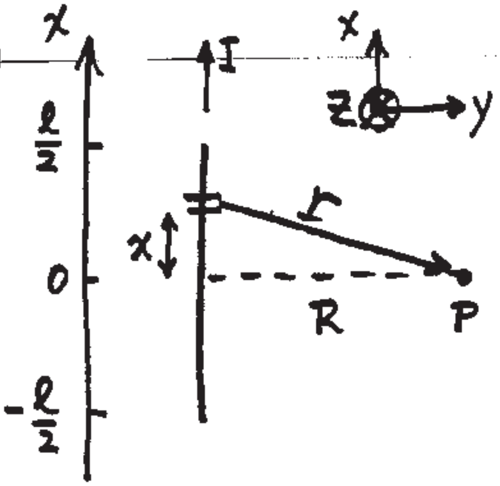
where $r = (l/2\pi N)$ is radius of each of N loops.

The area A of coil is then $A = \pi r^2 = \pi (l/2\pi N)^2 = \dfrac{l^2}{4\pi N^2}$

The torque $\tau = NIAB = NIB(l^2/4\pi N^2) = (Il^2 B/4\pi N)$

$$\boxed{\tau = (Il^2 B/4\pi N)}$$

(5)



Biot-Savart Law

$$d\underline{B} = \frac{\mu_0 I}{4\pi}\,\frac{d\underline{l}\times\underline{r}}{r^3}$$

$$x\hat{x} + \underline{r} = R\hat{y}$$

$$\underline{r} = -x\hat{x} + R\hat{y}\ ;\quad r^2 = (x^2+R^2)$$

$$d\underline{l} = (dx)\hat{x} \qquad r^3 = (x^2+R^2)^{3/2}$$

$$d\underline{B} = \frac{\mu_0 I}{4\pi}\,\frac{(dx)\hat{x}\times(-x\hat{x}+R\hat{y})}{(x^2+R^2)^{3/2}} = \frac{\mu_0 I R}{4\pi}\,\frac{dx\,(\hat{x}\times\hat{y})}{(x^2+R^2)^{3/2}}$$

Since $(\hat{x}\times\hat{y}) = \hat{z}$, $d\underline{B} = \left[\frac{\mu_0 I R}{4\pi}\,\frac{dx}{(x^2+R^2)^{3/2}}\right]\hat{z}$

Total magnetic field $\underline{B}$ at point P is,

$$\underline{B} = \int_{-l/2}^{l/2} d\underline{B} = \hat{z}\left[\frac{\mu_0 I R}{4\pi}\int_{-l/2}^{l/2}\frac{dx}{(x^2+R^2)^{3/2}}\right] = \hat{z}\left[\frac{\mu_0 I R}{4\pi}\left|\frac{x}{(x^2+R^2)^{1/2}}\right|_{-l/2}^{l/2}\right]$$

$$\underline{B} = \hat{z}\left[\frac{\mu_0 I R}{4\pi R^2}\left(\frac{2l}{2\sqrt{\frac{l^2+4R^2}{4}}}\right)\right] = \hat{z}\left[\frac{\mu_0 I\, l}{2\pi R}\,\frac{1}{\sqrt{l^2+4R^2}}\right]$$

$$\boxed{B = \frac{\mu_0 I}{2\pi R}\,\frac{l}{\sqrt{l^2+4R^2}}}$$

Physics S7B Solutions to Fourth Exam Aug. 21, 1992

(1)(a) Work done in infinitesimal volume change dV is

$$dW = p\,dV = \left(\frac{nRT}{V}\right)dV = 4RT\frac{dV}{V}$$

Total work $W = \int dW = 4RT\int_{V_i}^{2V_i}(dV/V) = 4RT\ln 2$

$$W = 4(8.31)(400)(0.693) \qquad \boxed{W = 9216 \text{ Joules}}$$

of work done by gas in isothermal expansion.

(b) For ideal gas at constant temperature: $\Delta U = 0 \Rightarrow Q = W$. Since expansion is reversible, $dS = (dQ/T)$ and, since $Q > 0$

$$\Delta S = \int dS = \int\frac{dQ}{T} = \frac{1}{T}\int dQ = \frac{Q}{T} = \frac{9216}{400} = 23\ JK^{-1} \qquad \boxed{\Delta S = 23\ JK^{-1}}$$

(2) Equation of state, with 1 mole, and $T = $ const.,

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT \implies p(V) = \frac{RT}{(V-b)} - \frac{a}{V^2}$$

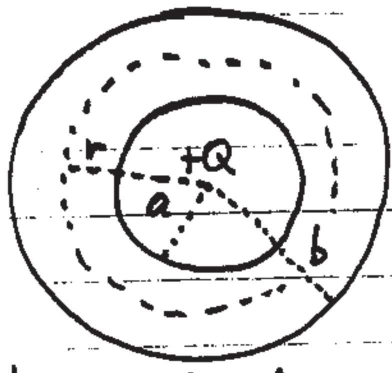
Work W done by gas is

$$W = \int_{V_i}^{V_f} p(V)\,dV = \int_{V_i}^{V_f}\left[\frac{RT}{(V-b)} - \frac{a}{V^2}\right]dV = RT\int_{V_i}^{V_f}\frac{dV}{(V-b)} - a\int_{V_i}^{V_f}\frac{dV}{V^2}$$

$$W = RT\Big[\ln(V-b)\Big]_{V_i}^{V_f} + a\left[\frac{1}{V}\right]_{V_i}^{V_f} = RT\ln\left(\frac{V_f - b}{V_i - b}\right) + a\left(\frac{1}{V_f} - \frac{1}{V_i}\right)$$

$$\boxed{W = RT\ln\left(\frac{V_f - b}{V_i - b}\right) + a\left(\frac{1}{V_f} - \frac{1}{V_i}\right)}$$ work done by gas

(3) Charge density $\rho(r) = Ar^{-1}$ for $a < r < b$; point charge $+Q$ at $r = 0$



To calculate $E(r)$, which we can assume spherically symmetric, use Gauss' Law. The Gaussian surface is a sphere of radius r centered at the point charge.

Let $q(r)$ be charge inside Gaussian sphere that is due to the charge density $\rho(r)$, and dq be element of this charge inside a spherical shell of radius r, thickness dr, and volume dV, where $dV = 4\pi r^2 dr$, so

$$dq = \rho(r)\left[4\pi r^2 dr\right] \Rightarrow dq = 4\pi A r^{-1} r^2 dr = 4\pi A r\, dr$$

$$q = \int_a^r dq = 4\pi A \int_a^r r\, dr = 4\pi A \left[\frac{r^2}{2}\right]_a^r = 2\pi A(r^2 - a^2)$$

Total charge Q' inside Gaussian sphere is $Q' = Q + q(r)$, so

$$Q' = Q + 2\pi A(r^2 - a^2)$$

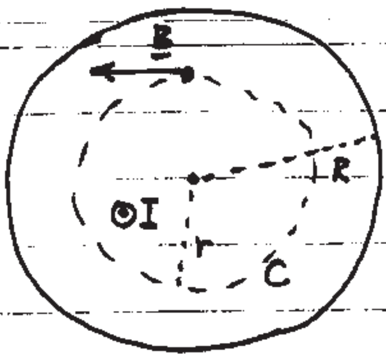
Gauss' Law: $\oint_{G.S.} E(r)\, dA = \frac{Q'}{\epsilon_0} \Rightarrow \epsilon_0 E(r) \oint dA = Q + 2\pi A(r^2 - a^2)$

$$\epsilon_0 E(r)\left[4\pi r^2\right] = \left[Q + 2\pi A(r^2 - a^2)\right] \Rightarrow \epsilon_0 E(r) = \frac{Q + 2\pi A(r^2 - a^2)}{4\pi r^2}$$

$$\epsilon_0 E(r) = \frac{Q}{4\pi r^2} + \frac{A}{2} - \frac{2\pi A a^2}{4\pi r^2}$$

$$\boxed{E(r) = \left(\frac{Q}{4\pi\epsilon_0} - \frac{Aa^2}{2\epsilon_0}\right)\frac{1}{r^2} + \frac{A}{2\epsilon_0}}$$ for $(a < r < b)$

(4)



Let R = radius of circular wire

Current density is

$$J = (I/\pi R^2)$$

Let $i(r)$ = current inside circle of radius r

Since current is spatially uniform,

$$i(r) = J(\pi r^2) = I(r^2/R^2)$$

From Ampere's Law: $\oint_C \underline{B} \cdot d\underline{l} = \mu_0 i(r)$ where $i(r)$ is current within C

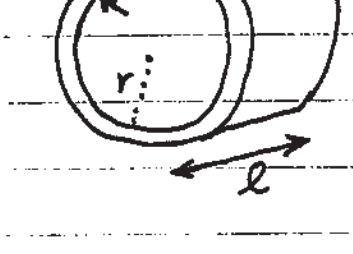
Since $\underline{B} \parallel d\underline{l}$, and assuming circular symmetry of $B(r)$, we have

$$\oint_C \underline{B} \cdot d\underline{l} = \mu_0 I (r^2/R^2) = B(r) \oint_C dl = B(r)[2\pi r] \Rightarrow \boxed{B(r) = (\mu_0 I / 2\pi R^2) r}$$

Magnetic energy density $u_B(r) = \frac{1}{2\mu_0}[B(r)]^2 = \frac{1}{2\mu_0}\left(\frac{\mu_0^2 I^2}{4\pi^2 R^4}\right) r^2$

To find total magnetic energy U_B within length l of wire

$dU_B = u_B\, dV$ where $dV = 2\pi r l\, dr$ is volume of cylindrical shell of length l, radius r, thickness dr

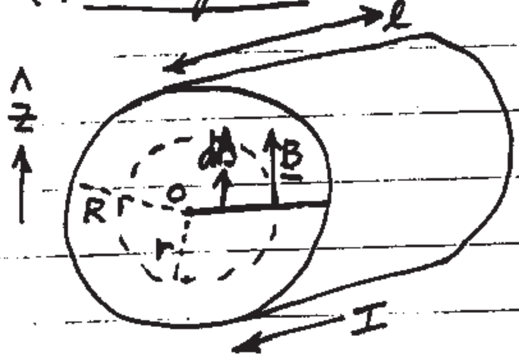


$$dU_B = \frac{1}{2\mu_0}\left(\frac{\mu_0^2 I^2}{4\pi^2 R^4}\right) r^2 (2\pi r l)\, dr = \frac{\mu_0 I^2 l}{4\pi R^4} r^3\, dr$$

$$U_B = \int_0^R dU_B = \frac{\mu_0 I^2 l}{4\pi R^4} \int_0^R r^3\, dr = \frac{\mu_0 I^2 l}{4\pi R^4}\left[\frac{R^4}{4}\right]$$

$\frac{U_B}{l}$ is magnetic energy per unit length; $\boxed{\frac{U_B}{l} = \frac{\mu_0 I^2}{16\pi}}$

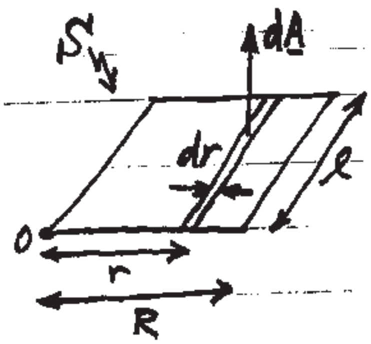
(3) <u>Diagram</u> is below: $\underline{B}(r) = \frac{\mu_0 \dot{i}(r)}{2\pi r}\hat{z}$



where $i(r)$ is current inside cylinder of radius r and length l; $\underline{B}(r)$ is field (due to $i(r)$) at distance r from center O of wire. Since current I is spatially uniform, $i(r) = I(r^2/R^2)$,

so $$B(r) = \frac{\mu_0 I r^2}{2\pi R^2 r} = (\mu_0 I/2\pi R^2)\, r$$

Let $d\underline{A}$ be vector element of area in plane S in drawing given, where



$$\left.\begin{aligned} d\underline{A} &= (dA)\hat{z} = (l\,dr)\hat{z} \\ \underline{B}(r) &= (\mu_0 I/2\pi R^2 r)\hat{z} \end{aligned}\right\} \underline{B}\cdot d\underline{A} = \frac{\mu_0 I l}{2\pi R^2}\, r\, dr$$

<u>Total</u> flux Φ through surface S is $\Phi = \int d\Phi$

$$\Phi = \int d\Phi = \int_0^R \underline{B}\cdot d\underline{A} = \frac{\mu_0 I l}{2\pi R^2}\int_0^R r\,dr = \frac{\mu_0 I l}{2\pi R^2}\left[\frac{R^2}{2}\right]$$

$$\boxed{\Phi = \frac{\mu_0 I l}{4\pi}}$$ is total flux through S