University of California at Berkeley
Physics 112, Fall 2000
Final Examination

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<u>Problem 1</u>
The subject of this problem is an ideal gas of N particles confined to a one-dimensional line segment of length L. The energy of orbital n is $E_n = C n^2 / L^2$, where C is a constant and n is an integer that is greater than ~~or equal to~~ 0.

(a) Evaluate the single-particle partition function by approximating the sum by an integral. A useful integral is the following: the integral from 0 to infinity of $\exp(-x^2)$ with respect to x is $(\pi^{1/2})/2$.

(b) Using the approximation that the N-particle partition function is given by $Z_N = (Z_1)^N / N!$, find the free energy F and from this, the entropy.

(a) $\quad Z_1 = \sum_{n=1}^{\infty} \exp\left(- C n^2/L^2 \tau\right)$

$\qquad \approx \int_{n=0}^{\infty} \exp\left(- C n^2/L^2 \tau\right) dn \quad = \left(\frac{C}{L^2 \tau}\right)^{-1/2} \int_0^{\infty} \exp(-x^2) dx$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = L\sqrt{\frac{\tau \pi}{C}} \, / 2$

(b) $F = -\tau \log Z_N \quad = \tau\left[\log(N!) - N \log Z_1\right]$

$\qquad = \tau\left[N \log N - N - N \log\left(n_{Q1} L\right)\right] , \quad n_{Q1} = \frac{1}{2}\sqrt{\frac{\tau \pi}{C}}$

$\qquad = \tau N\left[\log\left(\frac{N/L}{n_{Q1}}\right) - 1\right]$

$\sigma = -\left(\partial F/\partial \tau\right)_L = -N\left[\log\left(\frac{N/L}{n_{Q1}}\right) - 1\right] - \tau N\left[-\frac{\partial}{\partial \tau} \log\sqrt{\tau}\right]$

$\qquad = N\left[\log\left(\frac{n_{Q1}}{N/L}\right) + \frac{3}{2}\right]$

Problem 2

The subject of this problem is a one-dimensional version of thermal radiation. We consider a one-dimensional line segment of length L. The possible frequencies for electromagnetic modes are those for which L an integer multiple of a half-wavelength. Find the total energy at temperature T. Use: $\int_0^\infty u\, du / (e^u - 1) = \pi^2/6$

$$L = n \frac{\lambda}{2}$$

$$\lambda f = c \quad \Longleftrightarrow \quad \lambda \frac{\omega}{2\pi} = c$$

$$\omega = 2\pi c / \lambda = 2\pi c \left(\frac{n}{2L}\right) = \frac{n \pi c}{L}$$

energy:

$$U = 2 \sum_{n=1}^\infty \langle s(n) \rangle \hbar \omega_n \qquad \text{Factor of 2 for polarization states}$$

$$= 2 \sum_{n=1}^\infty \frac{\hbar \omega_n}{\exp(\hbar \omega_n / \tau) - 1} \qquad \text{using the Planck distribution function}$$

$$\approx 2 \int_0^\infty \frac{\hbar n \pi c / L}{\exp(\hbar n \pi c / L\tau) - 1}\, dn$$

$$u = \hbar n \pi c / L\tau, \quad \text{substitution}$$

$$= 2\tau \left(\frac{L\tau}{\hbar \pi c}\right) \int_0^\infty \frac{u\, du}{e^u - 1}$$

$$= 2 \frac{L \tau^2}{\hbar \pi c} \left(\frac{\pi^2}{6}\right)$$

$$= \frac{L \tau^2 \pi}{3 \hbar c}$$

Problem 3

Consider a lattice of fixed hydrogen atoms. Suppose that each atom can exist in three states:

State	Number of electrons	Energy
positive ion	0	E_0
neutral atom	1	E_1
negative ion	2	E_2

Given the chemical potential and the temperature, what is the average number of electrons per atom, $\langle N \rangle$? What is $\langle N^2 \rangle$?

Gibbs sum: $\quad \mathfrak{Z} = \lambda^0 \exp(-E_0/\tau) + \lambda^1 \exp(-E_1/\tau) + \lambda^2 \exp(-E_2/\tau)$

$$\langle N \rangle = \lambda \frac{\partial}{\partial \lambda} \log \mathfrak{Z} = \frac{\lambda \exp(-E_1/\tau) + 2\lambda^2 \exp(-E_2/\tau)}{\mathfrak{Z}}$$

Using Probability = Gibbs factor / $\mathfrak{Z}$, we have

$$\langle N^2 \rangle = \frac{\lambda \exp(-E_1/\tau) + 4\lambda^2 \exp(-E_2/\tau)}{\mathfrak{Z}}, \qquad \lambda = e^{\mu/\tau}$$

Problem 4

Consider an ideal monatomic gas in a two-dimensional space. There are N atoms in a square of area A = L². The temperature is T. You may use the facts that the single-particle states are identified by two indices n_x and n_y, each of which is greater than ~~or equal to~~ 0, and that the energy of such a state is E = C[$(n_x)^2$ + $(n_y)^2$]/L², where C is a constant.

(a) Find the chemical potential [Remember, use the classical limit for
the occupancy].
(b) Find the total energy.
(c) Find the entropy.

(a) $\quad N = \sum\limits_{n_x=1}^{\infty} \sum\limits_{n_y=1}^{\infty} f(n_x, n_y) = \sum\limits_{n_x=1}^{\infty} \sum\limits_{n_y=1}^{\infty} \lambda \exp\left(- C(n_x^2+n_y^2)/L^2\tau\right)$

approximate with an integral: (n = radius in n_x-n_y space)

$N = \dfrac{\pi}{2} \displaystyle\int_0^{\infty} \lambda \exp\left(-Cn^2/L^2\tau\right) n \, dn \qquad \left[dA = \dfrac{\pi}{2} n\,dn\right]$

$= \dfrac{\pi}{2} \lambda \left(\dfrac{C}{L^2\tau}\right)^{-1} \displaystyle\int_0^{\infty} e^{-x^2} x \, dx$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left[y = x^2, \; dy = 2x\,dx\right]$

$= \dfrac{\pi}{4} \lambda \dfrac{L^2\tau}{C} \displaystyle\int_0^{\infty} e^{-y} \, dy$

$N = \lambda A\, n_{Q2} \quad , \quad n_{Q2} = \dfrac{\pi\tau}{4C}$

$\mu = \tau \log \lambda = \tau \log\left(\dfrac{N/A}{n_{Q2}}\right)$

(b) $U = \sum\limits_{n} E_n f(E_n) = \sum\limits_{n} E_n \lambda \exp\left(-E_n/\tau\right)$

$= \lambda \tau^2 \dfrac{\partial}{\partial \tau} \sum exp\left(-E_n/\tau\right) = \lambda \tau^2 \dfrac{\partial}{\partial \tau}\left(\dfrac{N}{\lambda}\right)$

$= \lambda \tau^2 \dfrac{\partial}{\partial \tau}\left(A\, n_{Q2}\right) = \lambda \tau A\, n_{Q2} = N\tau$

(c) F is obtained from an integral of $\mu \, dN$, as in the 3-D case:

$F = N\tau \left[\log\left(\dfrac{N/A}{n_{Q2}}\right) - 1\right] . \qquad \sigma = -\left(\dfrac{\partial F}{\partial \tau}\right)_{A,N} = N\left[\log\left(\dfrac{n_{Q2}}{N/A}\right) + 2\right]$

Consider a system of N bosons of spin zero. Assume there are two single-particle orbitals, and they have energies are 0 and E. Find the temperature at which the occupation of the lower orbital is twice that of the higher orbital. Assume N >> 1.

Bose-Einstein distribution function:

$$f(\varepsilon, \tau) = \frac{1}{\exp[(\varepsilon - \mu)/\tau] - 1}$$

We have:

$$f(E, \tau) = \frac{1}{\exp[(E - \mu)/\tau] - 1}$$

$$f(0, \tau) = \frac{1}{\exp[-\mu/\tau] - 1}$$

Thus,

$$\exp[(E - \mu)/\tau] = 1 + \frac{1}{f(E, \tau)} \longleftarrow \frac{1}{3}N$$

$$\exp[-\mu/\tau] = 1 + \frac{1}{f(0, \tau)} \longleftarrow \frac{2}{3}N$$

Divide:

$$\exp(E/\tau) = \frac{1 + \frac{1}{N/3}}{1 + \frac{1}{2N/3}} = \frac{1 + \frac{3}{N}}{1 + \frac{3}{2N}} \approx 1 + \frac{3}{N} - \frac{3}{2N}$$

$$= 1 + \frac{3}{2N}$$

$$\frac{E}{\tau} \approx \frac{3}{2N} \qquad \text{Since} \quad N >> 1$$

$$\tau = \frac{2}{3} N \varepsilon$$

An ideal refrigerator is used to pump heat from an object to a high-temperature reservoir at temperature T_h. The heat capacity of the object is a function of temperature: $C = a\, T^b$, where a and b are positive constants. If the object is initially at temperature $T_i < T_h$, how much work is necessary to cool it to $T_i/2$?

Consider a small amount of work dW.

$$\frac{dQ_\ell}{dW} = \frac{T_\ell}{T_h - T_\ell}$$
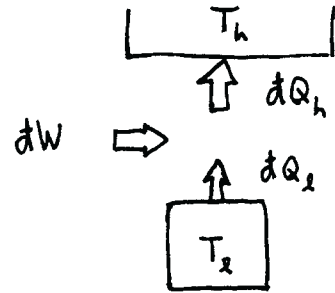
The change in temperature of the object is

$$dT_\ell = -\, dQ_\ell / C$$

$$= -\, \frac{dQ_\ell}{a T_\ell^b}$$

$$= -\, \frac{dW}{a T_\ell^b}\, \frac{T_\ell}{T_h - T_\ell}$$

$$dW = -\, a T_\ell^b \, dT_\ell \, \frac{T_h - T_\ell}{T_\ell}$$

$$W = \int_{T_i}^{T_i/2} dW = \int_{T_i/2}^{T_i} -\, dW$$

$$= \int_{T_i/2}^{T_i} a\, T_\ell^b \, \frac{T_h - T_\ell}{T_\ell}\, dT_\ell$$

$$= a \int_{T_i/2}^{T_i} \left[T_h\, T_\ell^{b-1} - T_\ell^b \right] dT_\ell$$

$$= a \left[T_h\, \frac{T_\ell^b}{b} - \frac{T_\ell^{b+1}}{b+1} \right]_{T_i/2}^{T_i}$$

$$= a \left[\frac{T_h}{b}\, T_i^b \left(1 - 2^{-b}\right) - \frac{1}{b+1}\, T_i^{b+1}\left(1 - 2^{-b-1}\right) \right]$$

Problem 7

In class, we calculated the pressure as a function of altitude for an isothermal atmosphere. Explain how this result was derived, and use it to estimate the pressure at an altitude of 5000m. Use the following approximate values (for the purposes of this problem, assume these numbers have three-digit accuracy):
the average mass of air molecule = 5×10^{-26}kg, T = 300K, g = 10m/s^2, 1atm = 100000Pa. The value of the Boltzmann constant is 1.38×10^{-23} J/K.

Use the above result to estimate the temperature of boiling water at 5000m. Use a value of 2300J/g for the heat of vaporization of water. Avogadro's number is 6.02×10^{23} mol^{-1}. Assume a water molecule has a mass of 18 times 1.67×10^{-27}kg.

Total chemical potential: $\mu = \tau \log(n/n_Q) + Mgh$

This is constant, as a function of height h. Thus,

$$\tau \log[n(h)/n_Q] + Mgh = \tau \log[n(0)/n_Q]$$

From this follows $p(h) = p(0) \exp(-Mgh/\tau)$.

$$= 5.47 \times 10^4 \text{ Pa}$$

Next, we use the Clausius - Clapyron equation and $\Delta v \approx v_g$. Also, we treat the gas as ideal. Thus,

$$\frac{dp}{dT} = \frac{Lp}{k_B T^2}$$

L = latent heat of vaporization per molecule

$$= (2300 \text{ J/g})(18 \times 1.67 \times 10^{-24} \text{ g})$$

T = 373 K for this calculation
$p \approx 8\times10^4$ Pa (approx midpoint)

$\Delta T \approx \Delta p / (dp/dT) \approx 20°K$.
$T_{5000} \approx 80°C$ (very approximate)

More accurate:
use $\log p = -L/\tau + $ const
$\Rightarrow \log(P_1/P_2) = -L\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)$
$\Rightarrow \frac{Mgh}{\tau_{atmosphere}} = L\left(\frac{1}{\tau_{5000}} - \frac{1}{\tau_0}\right)$
$\Rightarrow T_{5000} = 84°C$