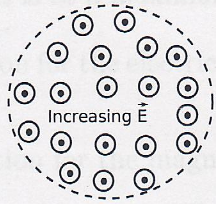
Problem 1 [20 pts]

This problem consists of short multiple choice questions and is mostly conceptual. Please put your answer in the space provided.



b (i) Consider an electric field which is coming out of the page and is restricted to a circular region. Say the magnitude of the $\vec{E}$-field is increasing, then the induced $\vec{B}$-field will point:

3pt

a) clockwise

b) counter-clockwise

c) into the page

d) out of the page

2pt

T (ii) True or False: Streams and lakes appear shallower than they actually are.

a (iii) If you place a two slit interference setup (laser, slits, and screen) under water, the pattern on the screen will:

3pt

(a) condense

(b) stretch out

(c) remain the same

3pt

F (iv) True or False: The image produced by a convex lens will always be a real image.

a (v) Compared to blue light, the single slit diffraction pattern of red light has a:

3pt

(a) wider central maximum

(b) narrower central maximum

(c) same size central maximum

b (vi) If you send unpolarized light of intensity I_0 through two polarizers which are oriented 180° with respect to each other, the resulting light has intensity:

3pt

(a) I_0

(b) 0.5 I_0

(c) 0.25 I_0

(d) 0

T (vii) True or False: It is impossible to have total internal reflection if the index of refraction of the fiber optic cable is less than that of the surrounding medium (for example, the cable has $n_{cable} = 1.2$, and it is placed in water $n_w = 1.33$).

3pt

Problem 3 [20 pts]

Consider a pair of convex lenses denoted by Lens$A$ and Lens$B$ aligned along an axis distance $L = f_A + f_B$ apart, where f_A and f_B are the focal lengths of the two lenses respectively. Now we have a parallel laser beam of width w_1 coming from the right to the two lenses system (see figure below).

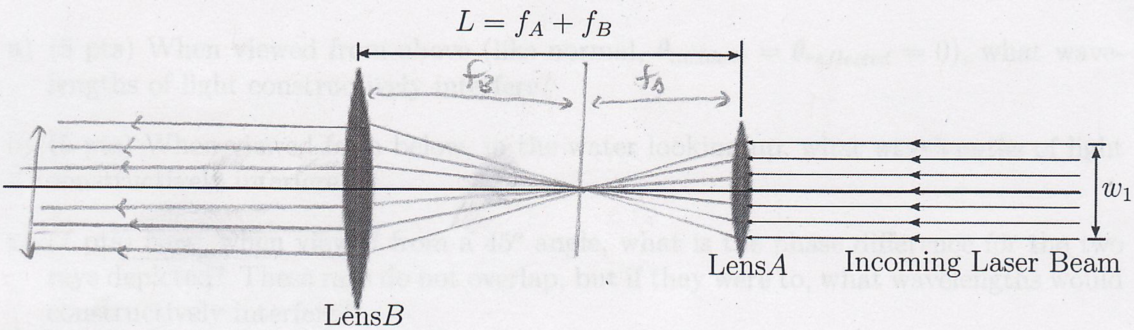
a) (4 pts) What must the object distance d_o be, if the initial rays enter Lens A completely parallel (*Hint: use the lens equation if you don't know off-hand*).

b) (6 pts) Do the ray diagram and find out the width w_2 of the outgoing laser beam. Note, the outgoing laser beams are again parallel to each other.

Now we replace the convex Lens$A$ by a concave LensA' of focal length $-f_{A'}$, where $f_{A'}$ is positive.

c) (5 pts) Find the new separation L' so that you get a parallel outgoing laser beam.

d) (5 pts) What's the new width w_2' of the outgoing laser beam?



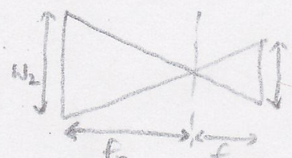
a) by thin lense eqn for Lens A

$\frac{1}{d_{oA}} + \frac{1}{d_{IA}} = \frac{1}{f_A}$ w/ $d_{oA} = \infty$ for parallel rays $\Rightarrow d_{IA} = f_A$ — 1 pt

then for Lens B $d_{oB} = L - d_{IA} = f_B$ — 1 pt

$\Rightarrow \frac{1}{d_{oB}} + \frac{1}{d_{IB}} = \frac{1}{f_B}$ while $d_{oB} = f_B \Rightarrow \frac{1}{d_{IB}} = 0$, i.e. $d_{IB} = \infty$. — 2pt.

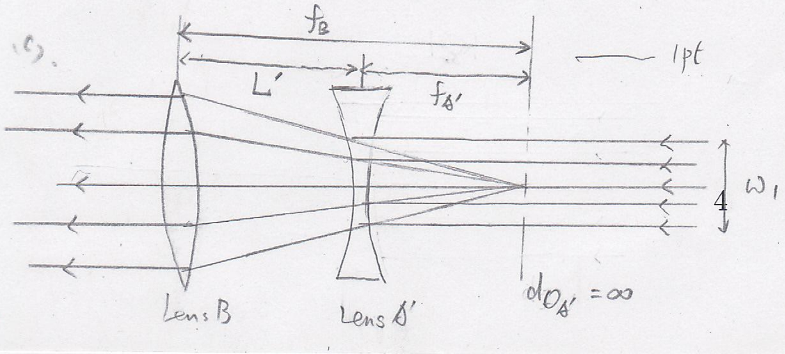
b) (see diagram above), we have similar triangle



3pt

$\Rightarrow \frac{w_2}{f_B} = \frac{w_1}{f_A}$ or $w_2 = \frac{f_B}{f_A} \cdot w_1$. — 3 pt

c)



for the parallel incoming rays. $d_{oA'} = \infty$

then. the thin lens eqn determines the virtual image distance $d_{IA'}$

$\frac{1}{d_{IA'}} + \frac{1}{d_{oA'}} = \frac{-1}{f_{A'}} \Rightarrow d_{IA'} = -f_{A'}$.

taking the virtual image as the object for

Lens B, we have the object distance $d_{O_B} = L' - d_{I_{A'}} = L' + f_{A'}$ (see figure).

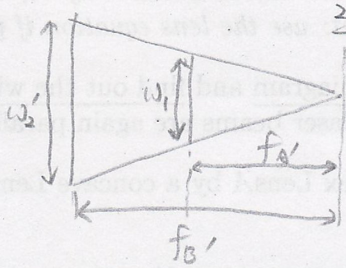
then, we have $\frac{1}{f_B} = \frac{1}{d_{O_B}} + \frac{1}{d_{I_B}} = \frac{1}{L'+f_{A'}} + \frac{1}{d_{I_B}}$ — 1pt

Requiring the outgoing rays being parallel $\Rightarrow d_{I_B} = \infty$ — 1pt

$\Rightarrow f_B = L' + f_{A'}$ or $L = f_B - f_A$. — 2pt

*

(d). again we have similar triangles 2pt



$\Rightarrow \frac{w_2'}{f_B} = \frac{w_1}{f_{A'}}$ or $w_2' = \frac{f_B}{f_{A'}} \cdot w_1$

— 3pt

Problem 5 [20 pts]

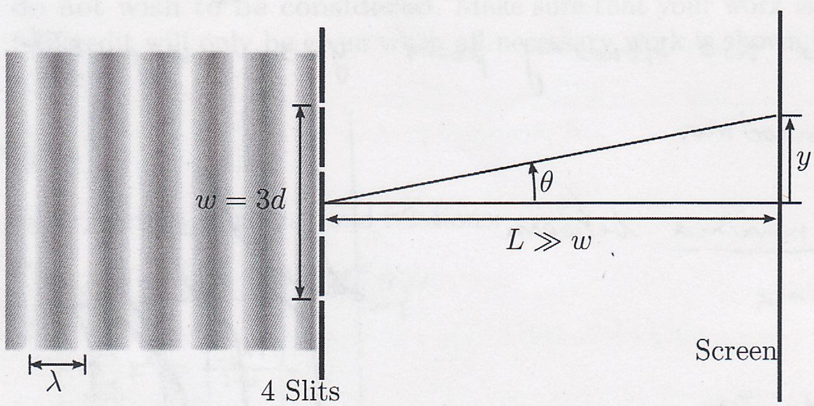
A plane-wave of wave number k ($k \equiv \frac{2\pi}{\lambda}$) comes to a plate with 4 thin slits, which has equal spacing d and hence the total spacing for the 4 slits is $w = 3d$ (see figure below). After passing the thin slits, the EM-wave creates an interference pattern on a screen distance L ($L \gg w$) away from the slits. For this problem, you may use the small angle approximation, where $\tan\theta \approx \sin\theta \approx \theta$.

a) (10 pts) Show that the ratio of the intensity at a height y (measured from the middle of the screen) to the intensity at the center of the screen has the form:

$$\frac{I(y)}{I_0} = \frac{1}{16}\left(\frac{\sin\left(2\frac{kdy}{L}\right)}{\sin\left(\frac{1}{2}\frac{kdy}{L}\right)}\right)^2$$

b) (2 pts) How many secondary maxima are there between the primary maxima (which are located at $d\sin\theta = m\lambda$, for integer m)?

c) (8 pts) Find the height y of the first four dark spots above the center spot.



2pt.

(a) using the complex electric field, we have $E_{tot}(\theta) = \mathrm{Re}\left(\sum_{l=0}^{3} E_0 e^{i[k(L\sec\theta + l\cdot d\cdot\sin\theta) - \omega t]}\right)$

since we want to find the relative intensity. one the magnitude of such a electric field matters. i.e. we only need to focus on the part:

$$\left|\sum_{l=0}^{3} e^{ikl\cdot d\sin\theta}\right| = \left|\frac{1-e^{4\cdot ik\cdot d\sin\theta}}{1-e^{ik\cdot d\sin\theta}}\right|$$ by the formula of geometric series on the front page.

$$= \left|\frac{-e^{\frac{4\cdot i\cdot k\cdot d\cdot\sin\theta}{2}}\left(e^{2\cdot i\cdot k\cdot d\cdot\sin\theta} - e^{-2ik\cdot d\cdot\sin\theta}\right)}{-e^{\frac{ik\cdot d\cdot\sin\theta}{2}}\left(e^{\frac{ik\cdot d}{2}\sin\theta} - e^{-i\frac{k\cdot d}{2}\sin\theta}\right)}\right| = \left|\frac{\sin(2\cdot k\cdot d\cdot\sin\theta)}{\sin(\frac{k\cdot d}{2}\cdot\sin\theta)}\right|$$ — 3pt

while for θ small. $\sin\theta \sim \tan\theta = y/L$ we have (the magnitude at y) $= \left|\frac{\sin(2\frac{k\cdot d\cdot y}{L})}{\sin(\frac{k\cdot d\cdot y}{2L})}\right|$

$$\Rightarrow \frac{I(y)}{I(0)} = \frac{(\text{Amplitude at } y)^2}{(\text{Amplitude at } 0)^2} = \frac{\left(\sin 2\cdot\frac{k\cdot d\cdot y}{L} \Big/ \sin\frac{k\cdot d\cdot y}{2L}\right)^2}{\lim_{\epsilon\to 0}\left(\sin 2\cdot\frac{k\cdot d\cdot\epsilon}{L} \Big/ \sin\frac{k\cdot d\cdot\epsilon}{2L}\right)^2} = \frac{1}{4^2}\cdot\underbrace{\frac{\left(\sin 2\cdot\frac{k\cdot d\cdot y}{L}\right)^2}{\left(\sin\frac{k\cdot d\cdot y}{2L}\right)^2}}_{3pt}$$

2pt.

(b) the maxima of the interference pattern appears when the amplitude of the electric field has a local extremeum, that is

$$0 = \frac{d}{dy}\left(\frac{\sin\frac{2kd\cdot y}{L}}{\sin\frac{k\cdot d\cdot y}{2L}}\right) = \frac{1}{\sin^2\left(\frac{kd\cdot y}{2L}\right)}\left\{\frac{2\cdot k\cdot d}{L}\cos\left(\frac{2kd}{L}y\right)\sin\left(\frac{k\cdot d}{2L}y\right) - \frac{k\cdot d}{2L}\sin\left(\frac{2\cdot k\cdot d}{L}y\right)\cos\left(\frac{k\cdot d}{2L}y\right)\right\}$$

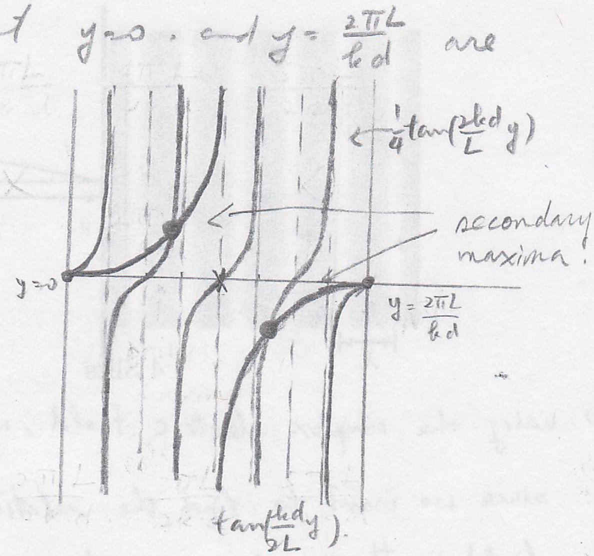
i.e. we seek for the location s.t. $\frac{2kd}{L}\cos\left(\frac{2kd}{L}y\right)\sin\left(\frac{k\cdot d}{2L}y\right) = \frac{k\cdot d}{2L}\sin\left(\frac{2kd}{L}y\right)\cos\left(\frac{kd}{2L}y\right) = 0$.

within half period of $\sin\frac{k\cdot d}{2L}y$ i.e. $y \in \left(0, \frac{2L\pi}{kd}\right)$ (the boundary will be the primary maxima)

the equation can be written as

$$\frac{\cos\left(\frac{2kd}{L}y\right)\sin\left(\frac{kd}{2L}y\right)}{\sin\left(\frac{2kd}{L}y\right)\cos\left(\frac{kd}{2L}y\right)} = \frac{\frac{kd}{2L}}{\frac{2kd}{L}} = \frac{1}{4} \quad \text{i.e.} \quad \tan\left(\frac{kd}{2L}y\right) = \frac{1}{4}\tan\left(\frac{2kd}{L}y\right)$$

and we are looking for sol's in one period of $\sin^2\left(\frac{kd}{2L}y\right)$. which is $y \in \left(0, \frac{2\pi L}{kd}\right)$. Note, the two boundary point $y=0$ and $y=\frac{2\pi L}{kd}$ are the location of primary maxima



$\Rightarrow$ there are <u>two secondary maxima</u> between a pair of primary maxima

c) the dark spots locate at $I(y) = 0$

i.e. $\sin\left(\frac{2kd}{L}y\right) = 0$ i.e. $\frac{2kd}{L}y \in$ integers

but at the zeros of $\sin\frac{kd}{2L}y$, it will be instead a primary maxima

$\Rightarrow$ zeros of $\sin\left(\frac{2kd}{L}y\right)$: $y = 0, \frac{\pi L}{2kd}, \frac{2\pi L}{2kd}, \frac{3\pi L}{2kd}, \frac{4\pi L}{2kd}, \frac{5\pi L}{2kd}$

$\sin\left(\frac{kd}{2L}y\right)$: $y = 0$, X, X, X, $\frac{2\pi L}{kd}$, X

primary maxima, dark spots, primary maxima, dark spot

$\Rightarrow$ dark spots at $y = \frac{\pi L}{2kd}, \frac{2\pi L}{2kd}, \frac{3\pi L}{2kd}, \frac{5\pi L}{2kd}, \ldots$

or $\frac{\lambda L}{4d}, \frac{2\lambda L}{4d}, \frac{3\lambda L}{4d}, \frac{5\lambda L}{4d}, \ldots$

(2pt each)

7C Midterm 1 Solutions

1 Problem 2

An electromagnetic wave is travelling in the $+y$ direction. The wave is linearly polarized in the $+z$ direction, has an amplitude of $50.0\frac{V}{m}$ and a wavelength of 30.0 nm. At $t = 0$ and $y = 0$, the electric field is at a maximum and points in the $+z$ direction.

1.1 Part a

Write down the equation for the electric field.

$$\begin{aligned} E &= 50.0\frac{V}{m}\cos(ky - \omega t)\hat{z} \\ k &= \frac{2\pi}{30.0nm} = 2.09 \times 10^{8} m^{-1} \\ \omega &= \frac{2\pi c}{\lambda} = 6.28 \times 10^{16}\frac{rad}{s} \end{aligned}$$

The amplitude is given. It is a cosine so that plugging in 0 gives a maximum. The $\hat{z}$ gives the polarization direction.

1.2 Part b

Write down the equation for the magnetic field for this wave.

$$\begin{aligned} E_0 &= cB_0 \\ B_0 &= 1.67 \times 10^{-7} T \\ B &= 1.67 \times 10^{-7} T \cos(ky - \omega t)\hat{x} \end{aligned}$$

The B field points in the +x direction so that $E \times B$ which is proportional to $\hat{z} \times \hat{x} = \hat{y}$ points in the direction of propogation.

1.3 Part c

Write down the Poynting vector as well as the intensity (the time averaged magnitude of S)

$$\begin{aligned}
S &= \frac{1}{\mu_0} E \times B \\
&= 6.64 \frac{W}{m^2} \cos^2(ky - \omega t)\hat{y} \\
I &= \langle |S| \rangle = 6.64 \frac{W}{m^2} \frac{1}{T} \int_0^T \cos^2(ky - \omega t) = 3.32 \frac{W}{m^2} \\
I &= \frac{1}{2} \epsilon_0 c E_0^2 = 3.318 \frac{W}{m^2}
\end{aligned}$$

Both formulas for I agree.

1.4 Part d

If the wave strikes a perfectly reflecting mirror ($A = 10m^2$, $m = 20kg$) square-on, what would be the acceleration of the mirror?

$$\begin{aligned}
P &= \frac{I}{c} \\
F &= 2PA \\
a &= \frac{F}{m} = \frac{2IA}{mc} = 1.11 \times 10^{-8} \frac{m}{s^2} \hat{y}
\end{aligned}$$

The force is multiplied by 2 as compared to a black surface because the light changes from having momentum in the $+y$ direction to the $-y$ direction rather than $+y$ direction to none at all.

2 Problem 4

2.1 Part a

The first ray gets a π phase shift upon reflection by the oil.
The second ray gets one upon reflection by the water. It also travels an extra $2t$ distance.

$$\begin{aligned}
\phi_1 &= \pi \\
\phi_2 &= \pi + 2t\frac{2\pi}{\lambda_{oil}} \\
\Delta\phi &= \frac{4\pi t n_{oil}}{\lambda} = 2\pi m \\
\lambda &= \frac{2tn_{oil}}{m}
\end{aligned}$$

This is vacuum wavelength for constructive interference.

2.2 Part b

The ray that travels straight through picks up a phase from travelling a distance t through the oil while the second ray gets phase from the 3t of oil as well as the reflection from the oil water surface. It does not get any from the oil air reflection.

$$\begin{aligned}
\phi_1 &= \frac{t2\pi}{\lambda_{oil}} \\
\phi_2 &= \pi + \frac{3t2\pi}{\lambda_{oil}} \\
\Delta\phi &= \frac{2t2\pi}{\lambda_{oil}} + \pi = 2m\pi \\
\frac{4n_{oil}t}{\lambda} + 1 &= 2m \\
\lambda &= \frac{4tn_{oil}}{2m-1} = \frac{2tn_{oil}}{m-\frac{1}{2}}
\end{aligned}$$

We could also see this by replacing m in the formula from a. This is because a constructive interference for part a would be destructive for part b and vice versa.

2.3 Part c

First we find the angle of refraction

$$\begin{aligned}
n_{air}\sin\frac{\pi}{4} &= n_{oil}\sin\theta \\
\sin\theta &= \frac{n_{air}}{\sqrt{2}n_{oil}}
\end{aligned}$$

$$
\begin{aligned}
\cos\theta &= \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{n_{air}^2}{2n_{oil}^2}} \\
&= \frac{1}{\sqrt{2}n_{oil}}\sqrt{2n_{oil}^2 - n_{air}^2}
\end{aligned}
$$

The first ray travels an extra distance in the air because of the separation between the beams (Call that x causing an extra $x\sin\frac{\pi}{4}$ in air) and gets a phase shift upon reflection by the oil. The second ray travels a distance $\frac{2t}{\cos\theta}$ inside the oil, it also gets a phase shift upon reflection from the water.

$$
\begin{aligned}
\tan\theta &= \frac{x}{2t} \\
x &= 2t\tan\theta \\
\phi_1 &= \pi + \frac{x\sin\frac{\pi}{4}2\pi n_{air}}{\lambda} \\
\phi_2 &= \pi + \frac{2t2\pi n_{oil}}{\cos\theta\lambda} \\
\Delta\phi &= \frac{4\pi t n_{oil}}{\cos\theta\lambda} - \frac{\pi\sqrt{2}2tn_{air}\tan\theta}{\lambda} \\
&= \frac{4\pi t n_{oil} - \pi 2\sqrt{2}n_{air}t\sin\theta}{\lambda\cos\theta} \\
&= \frac{4\pi t n_{oil} - \pi 2t\frac{n_{air}^2}{n_{oil}}}{\lambda\cos\theta} \\
&= \frac{2\pi t}{n_{oil}}\frac{2n_{oil}^2 - n_{air}^2}{\lambda\cos\theta} \\
&= \frac{2\pi t\sqrt{2}}{\lambda}\sqrt{2n_{oil^2} - n_{air}^2} \\
\Delta\phi &= 2\pi m \\
\lambda &= \frac{t\sqrt{2}}{m}\sqrt{2n_{oil}^2 - n_{air}^2}
\end{aligned}
$$

2.4 Part d

The largest wavelength for each part is given by plugging in $m = 1$ to each of the above formulae.

$$
\begin{aligned}
\frac{2tn_{oil}}{m} &= 480\mu m \\
\frac{4tn_{oil}}{2m-1} &= 960\mu m
\end{aligned}
$$

$$\frac{t\sqrt{2}}{m}\sqrt{2n_{oil}^2 - n_{air}^2} \quad = \quad 388\mu m$$