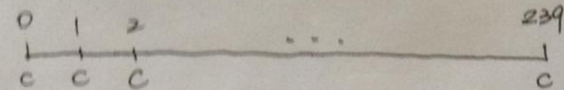
1. Solution:

<a>. The time line is:



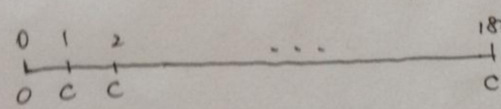
The effective monthly rate is $r = \frac{APR}{12}$ (since it is compounded monthly)

Hence, we have

$$150{,}000 = C + \frac{C}{1+r} + \cdots + \frac{C}{(1+r)^{239}}$$

$$\Rightarrow C = 985.826.$$

<b>. After 60 installments, there are 180 periods remaining on the loan,



The outstanding balance is the present value of these remaining payments, which is

$$B = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^{180}} = \frac{C}{r}\left[1 - \frac{1}{(1+r)^{180}}\right]$$

$$= 124{,}662.866.$$

<c>. Let n be the additional months needed, then

$$B = \frac{C}{1+r_1} + \frac{C}{(1+r_1)^2} + \cdots + \frac{C}{(1+r_1)^n},$$

where $r_1 = \frac{6\%}{12} = 0.5\%$, ($B$, C are given in (a), (b))

Solving for n, we get

$$n \doteq 200.5844.$$

2. Solution:

<a>. Let R_A, R_B be the rate of return for stock A, B, respectively. then we have

Condition	R_A	R_B	probability
Good	30%	40%	$\frac{1}{3}$
Average	5%	-7%	$\frac{1}{3}$
Bad	-20%	-15%	$\frac{1}{3}$

Therefore, we obtain

$$\begin{cases} r_A = E[R_A] = \frac{1}{3}(30\% + 5\% - 20\%) = 0.05 \\ \sigma_A^2 = var(R_A) = E[R_A^2] - (E[R_A])^2 = 0.04167 \end{cases}$$

and

$$\begin{cases} r_B = E[R_B] = \frac{1}{3}(40\% - 7\% - 15\%) = 0.06 \\ \sigma_B^2 = var(R_B) = E[R_B^2] - (E[R_B])^2 = 0.058867 \end{cases}$$

$$\sigma_{AB} = cov(R_A, R_B) = E[R_A \cdot R_B] - E[R_A] \cdot E[R_B] = 0.04583 .$$

(Note, you should show your work for computing σ_A^2, σ_B^2 and σ_{AB})

<b>. From part (a), we have

$$V = \begin{bmatrix} 0.04167 & 0.04583 \\ 0.04583 & 0.058867 \end{bmatrix}, \quad \tilde{r} = \begin{pmatrix} r_A - r_0 \\ r_B - r_0 \end{pmatrix} = \begin{pmatrix} 0.03 \\ 0.04 \end{pmatrix}$$

Therefore, the market portfolio is:

$$X(\rho_M) = \frac{V^{-1}\tilde{r}}{e^T V^{-1}\tilde{r}} = \begin{pmatrix} -0.2990 \\ 1.2990 \end{pmatrix}, \quad \left(e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

and the expected return for the market portfolio:

$$\rho_M = r_0 + \frac{\tilde{r}^T V^{-1} \tilde{r}}{e^T V^{-1} \tilde{r}} = 0.06299 .$$

(Alternatively, $\rho_M = \begin{pmatrix} r_A \\ r_B \end{pmatrix}^T X(\rho_M) = r_A(-0.2990) + r_B \cdot (1.2990) = 0.06299$).

<c>. $\rho = 0.04$, then

$$X(\rho) = \frac{\tilde{\rho}}{\rho_m} X(\rho_m) = \begin{pmatrix} -0.139105 \\ 0.6043392 \end{pmatrix}$$

and

$$X_0(\rho) = 1 - \frac{\tilde{\rho}}{\tilde{\rho}_m} = 0.5347755 .$$

<3>. Solution:

<a>. Using the security market line, we have

$$\begin{cases} r_A = r_0 + \beta_A (\rho_M - r_0) \\ r_B = r_0 + \beta_B (\rho_M - r_0) \\ r_A = 1.2 r_B \\ \beta_A = 1.25 \beta_B \end{cases} \Rightarrow \begin{cases} r_A = 0.24 \\ r_B = 0.2 \\ \beta_A = 2.5 \\ \beta_B = 2. \end{cases}$$

<b>. See part (a).

<c>. Suppose out portfolio is $(\overset{A}{\alpha}, \overset{B}{1-\alpha})$, then the variance of the portfolio is

$$f(\alpha) = var(\alpha R_A + (1-\alpha) R_B) = \alpha^2 \cdot \sigma_A^2 + (1-\alpha)^2 \sigma_B^2 + 2\alpha(1-\alpha)\sigma_{AB}$$

Setting $f'(\alpha) = 0$, we obtain $\alpha = 0.28$ (verify this!)

Thus, the portfolio should be $(0.28, 0.72)$.

<d>. We have

$$V = \begin{bmatrix} 0.36 & 0.18 \\ 0.18 & 0.25 \end{bmatrix}, \quad \tilde{r} = \begin{pmatrix} r_A - r_0 \\ r_B - r_0 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.16 \end{pmatrix}$$

$$\Rightarrow X(\rho) = X(30\%) = \frac{\tilde{\rho}}{\tilde{r}^T V^{-1} \tilde{r}} V^{-1} \tilde{r} = \begin{pmatrix} 0.7162162 \\ 0.729729 \end{pmatrix}$$

and $X_0(\rho) = -0.445945$.

4. Solution:

<a>. From your friend, we know the market portfolio consists of equal weights proportion of stock A and stock B.

Note that your friend is investing $\frac{600}{3000} = \frac{1}{5}$ in the risk-free asset, and $\frac{2400}{3000} = \frac{4}{5}$ in the market portfolio. Since she gets back 3300, this is a return of $\frac{3300-3000}{3000} = 10\%$. (the expected return of her portfolio).

Therefore,

$$0.10 = E\left[\tfrac{1}{5} r_0 + \tfrac{4}{5} R_M\right], \text{ with } r_0 = 0.02, \text{ we get}$$

$$\rho_M = E[R_M] = 0.12.$$

We also know $R_M = \frac{1}{2} R_A + \frac{1}{2} R_B$ (from our first observation), it follows that

$$\sigma_M^2 = var(R_M) = \tfrac{1}{4}\sigma_A^2 + \tfrac{1}{4}\sigma_B^2 + 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot cov(R_A, R_B)$$

$$= 0.0325.$$

<b>. Note that you want a 20% return, hence the fraction α to invest in the risk-free asset must satisfy:

$$0.2 = \underset{\substack{\downarrow \\ r_0}}{0.02}\cdot\alpha + (1-\alpha)\cdot\underset{\substack{\downarrow \\ \rho_M}}{0.12} \Rightarrow \alpha = -\tfrac{4}{5},\ 1-\alpha = \tfrac{9}{5}.$$

Therefore, we invest $-\frac{4}{5} \times 1000 = -800$ in risk-free asset (borrow)

invest $\frac{9}{5} \times 1000 \times \frac{1}{2} = 900$ in stock A

invest $\frac{9}{5} \times 1000 \times \frac{1}{2} = 900$ in stock B.

Since our portfolio can be written as $R_P = -\frac{4}{5} r_0 + \frac{9}{5} R_M$,

so $\sigma_P^2 = \left(\frac{9}{5}\right)^2 \cdot \sigma_M^2 = 0.1053$.

<c>. Note that

$$R_M = \frac{1}{2} R_A + \frac{1}{2} R_B$$

$$\Rightarrow \quad 1 = \beta_M = \frac{1}{2}\beta_A + \frac{1}{2}\beta_B$$

$$\Rightarrow \quad \beta_A = 1.3 \qquad (\text{ since } \beta_B = 0.7 \text{ })$$

(Alternatively, you can use the SML to get β_A).

<d>. We know $\beta_A = 1.3$, so $cov(R_A, R_M) = \beta_A \cdot \sigma_M^2 = 0.04225$.

Since $R_p = -\frac{4}{5} r_0 + \frac{9}{5} R_M$,

So $cov(R_p, R_M) = \frac{9}{5}\sigma_M^2$.

5. Solution:

Let R_i be the value of bet i, then

$$\begin{cases} E[R_1] = -9 \cdot \frac{1}{11} + 1 \cdot \frac{10}{11} = \frac{1}{11} \\ var(R_1) = E[R_1^2] - (E[R_1])^2 = (-9)^2 \cdot \frac{1}{11} + 1^2 \cdot \frac{10}{11} - (\frac{1}{11})^2 = \frac{1000}{121}. \end{cases}$$

$$\begin{cases} E[R_2] = -1 \cdot \frac{5}{11} + 1 \cdot \frac{6}{11} = \frac{1}{11} \\ var(R_2) = E[R_2^2] - (E[R_2])^2 = (-1)^2 \cdot \frac{5}{11} + (1)^2 \cdot \frac{6}{11} - (\frac{1}{11})^2 = \frac{120}{121}. \end{cases}$$

Similarly, $E[R_3] = \frac{1}{11}$ and $var(R_3) = \frac{120}{121}$

<b>. The covariance between bet 1 and bet 2 is

$$\begin{aligned} cov(R_1, R_2) &= E[R_1 R_2] - E[R_1] \cdot E[R_2] \\ &= (-9) \cdot (+1) \frac{1}{11} + (+1) \cdot (-1) \frac{5}{11} + (+1) \cdot (+1) \frac{5}{11} - (\frac{1}{11})^2 \\ &= -\frac{100}{121}. \end{aligned}$$

Suppose we allocate α to bet 1, and $(1-\alpha)$ to bet 2, the return of our bet is then $R = \alpha \cdot R_1 + (1-\alpha) R_2$.

$$\min_{\alpha} var(\alpha R_1 + (1-\alpha) R_2) \Rightarrow \alpha = \frac{1}{6}, \quad 1-\alpha = 1 - \frac{1}{6} = \frac{5}{6}.$$

<C>. Sure! The strategy is:

place $\frac{1}{11}$ of what you have in bet 1, $\frac{5}{11}$ of what you have in bet 2, and $\frac{5}{11}$ of what you have in bet 3.

$\boxed{6}$. Solution:

<a>. True. Note that for $YTM \le 15\%$, $\frac{50}{(1+\frac{YTM}{2})^i} \le \frac{200}{(1+YTM)^i}$, $i = 1, \dots, 20$.

Thus, $P_A = \frac{50}{1+\frac{YTM}{2}} + \cdots + \frac{50+1000}{(1+\frac{YTM}{2})^{20}} \le \frac{200}{1+YTM} + \cdots + \frac{200+4000}{(1+YTM)^{20}} = P_B$.

<b>. False.

We need to evaluate the NPV:

$$NPV = \sum_{i=1}^{100} i \left(\frac{-1}{1+0.1}\right)^i \doteq -0.2459595 < 0$$

(using the formula,

$$\sum_{i=1}^{n} i x^n = \frac{n x^{n+2} - (n+1) x^{n+1} + x}{(1-x)^2}.$$)

<c>. True.

$$r_i = r_0 + \beta_i(\rho_M - r_0) = r_0 + \frac{\sigma_{iM}}{\sigma_M^2}(\rho_M - r_0)$$
$$= r_0 + 0 = r_0$$

<d>. False. See ~~your~~ problem 4 in your HW5 for a counterexample.

(We can also see this geometrically.)