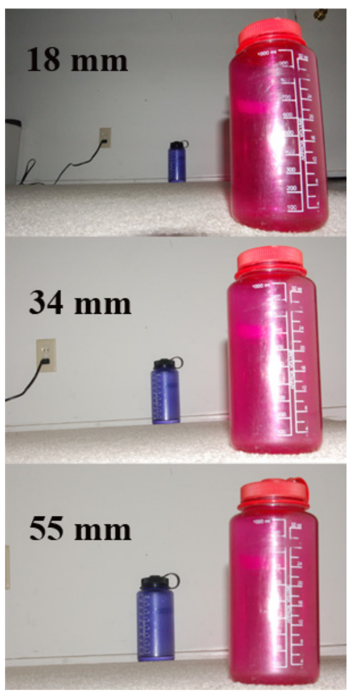
Physics H7C
Midterm, October 9^{th} 2012, 9:30-11:00

Note: *You are allowed one handwritten formula card (3.5" × 5" double sided). No calculators or any other electronic devices are permitted. Show all work, and take particular care to explain what you are doing. Please use the symbols described in the problems, define any new symbols that you introduce, and label any drawings that you make.*

1.

The figure below shows the image of two objects, of the same height but at different distances from the lens, using lenses with different focal lengths.



a. Using the thin lens equation,

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \tag{1}$$

write down an equation which explains why it appears as though the background water bottle is further away from the foreground bottle as the focal length is decreased. Assume that as the focal length is changed the camera is moved so that the size of the red water bottle is kept the same in each case, but that the distance, d, between the two water bottles is fixed.

Solution We are told that the foreground water bottle has the same image size even as the focal length of the lens changes, and that the distance between the foreground bottle and the background bottle is fixed at a value d. The size of the image of the foreground bottle is given by

$$h_i^{\text{fore}} = h_o^{\text{fore}} M_T^{\text{fore}} = h_o^{\text{fore}} \frac{s_i^{\text{fore}}}{s_o^{\text{fore}}}. \tag{2}$$

(where we have taken the absolute value of the transverse magnification, since we are only interested in the size of the image, not its orientation) The thin lens equation tells us

$$\frac{1}{s_o^{\text{fore}}} + \frac{1}{s_i^{\text{fore}}} = \frac{1}{f} \rightarrow s_i^{\text{fore}} = \frac{s_o^{\text{fore}} f}{s_o^{\text{fore}} - f}. \tag{3}$$

With this the size of the image of the foreground bottle is

$$h_i^{\text{fore}} = h_o^{\text{fore}} \frac{f}{s_o^{\text{fore}} - f}. \tag{4}$$

Since the size of the image remains the same as we change the focal length, the object distance must change as

$$s_o^{\text{fore}} = \frac{h_o^{\text{fore}}}{h_i^{\text{fore}}} f + f = f\left(1 + \frac{1}{M_T^{\text{fore}}}\right). \tag{5}$$

Therefore, in each image, the distance between the lens and the background water bottle is

$$s_o^{\text{back}} = s_o^{\text{fore}} + d = f\left(1 + \frac{1}{M_T^{\text{fore}}}\right) + d. \tag{6}$$

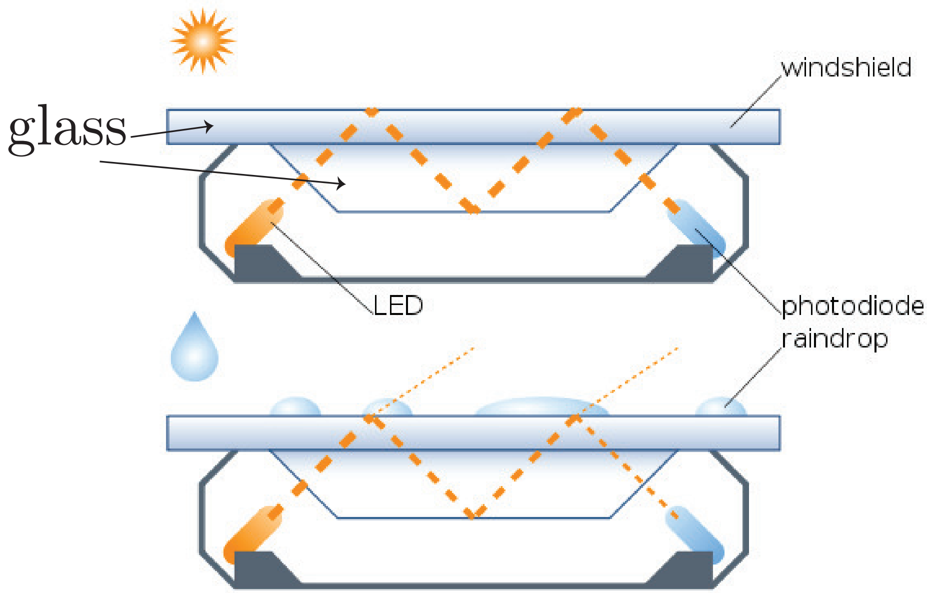
Just as in the case of the foreground bottle, the image size of the background bottle is

$$h_i^{\text{back}} = h_o^{\text{back}} \frac{f}{s_o^{\text{back}} - f} = h_o^{\text{back}} \frac{f}{f\left(1 + \frac{1}{M_T^{\text{fore}}}\right) + d - f} = h_o^{\text{back}} \frac{f}{f/M_T^{\text{fore}} + d} \tag{7}$$

Now everything in this equation is fixed except the focal length, f. This equation shows that if we decrease f while keeping the image size of the foreground bottle fixed, then the size of the background image will decrease relative to the size of the foreground bottle.

b. Total internal reflection is used to trigger automatic windshield wipers. A beam of light is directed from the top of the dashboard to the windshield at an angle so that when the windshield is dry total internal reflection causes it to be reflected to a detector on the other side of the car; when it is wet, most of the infrared beam is transmitted and does not get detected, signaling the windshield wipers to turn on. Write down an equation which determines the range of allowed incident angles for the initial beam. Take the index of refractions that are involved to be: $n_{\text{air}} = 1$, $n_{\text{glass}} = 3/2$, and $n_{\text{H}_2\text{O}} = 4/3$.

Solution



Total internal reflection occurs when $\sin(\theta) > n_t/n_i$. Since we want total internal reflection to occur when the windshield is dry we have the condition on the incident light on the left:

$$\sin(\theta) > n_{\text{air}}/n_{\text{glass}} = 1/(3/2) = 2/3. \tag{8}$$

We then want light to be transmitted when the windshield is wet- so that

$$\sin(\theta) < n_{\text{H}_2\text{O}}/n_{\text{glass}} = (4/3)/(3/2) = 8/9. \tag{9}$$

Putting these together, for the trigger to work, the angle must be between

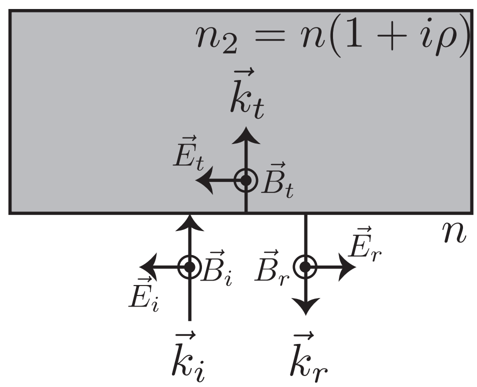
$$2/3 < \sin(\theta) < 8/9. \tag{10}$$

c. At the polarization (or Brewster's) angle all of the reflected light is polarized in a plane perpendicular to the incident plane. Could the trigger depend instead on the polarization of the light? If so, give a brief description of how this might work. *Hint: the polarization angle occurs when* $\theta_i + \theta_t = \pi/2$. Also address (again, briefly) whether you think this is a practical way to build such a trigger system.

Solution Yes it could. The initial beam would have to be very precisely aimed so that it hit the windshield *exactly* at the polarization angle- $\tan(\theta_p) = n_t/n_i = 1/(3/2) = 2/3$. Then, when the windshield became wet the beam would no longer be polarized. The receiver would have to measure the polarization that it receives- one way for this to occur is to put a polarizer in front which blocks all light which is polarized parallel to the glass surface (which is the only light that would reflect at the polarization angle). Then the trigger would set off the windshield wipers when it detected light (i.e., when some of the light that it receives is polarized perpendicular to the glass surface).

This would not be practical since it would have to be set up very precisely. Also, the amount of light that would get to the trigger would be very small since most of it would be transmitted through the windshield.

2. a. A plane electromagnetic wave traveling in a dielectric medium of index of refraction



n is reflected at normal incidence from the surface of a conductor. Find the phase change undergone by the reflected wave's electric field if the refractive index of the conductor is $n_2 = n(1 + i\rho)$. Assume that the materials have the same constant of permeability, $\mu = \mu_0$. *Hint: you will need to use the fact that* $1/(a + bi) = (a - bi)/(a^2 + b^2)$. *Also remember that* $a + bi = \sqrt{a^2 + b^2}e^{i\arctan(b/a)}$.

Solution

For normal incidence both the electric and magnetic fields are parallel to the interface. Faraday's equation and the Maxwell-Ampere equation both tell us that these components of the electric and magnetic fields are equal across the boundary:

$$\vec{E}_1^{||} = \vec{E}_2^{||} \tag{11}$$
$$\vec{B}_1^{||} = \vec{B}_2^{||}. \tag{12}$$

We can use this, along with the directions for the incident, reflected and transmitted electromagnetic fields to write

$$-\vec{E}_i + \vec{E}_r = -\vec{E}_t, \tag{13}$$
$$\vec{B}_i + \vec{B}_r = \vec{B}_t. \tag{14}$$

We want to convert the last equation from an equation which relates the magnetic fields to an equation which relates the electric field. We only need to remember that Faraday's equation tells us that for a plane wave:

$$\vec{k} \times \vec{E} = \omega\vec{B}. \tag{15}$$

Given that all three waves point in the z-direction, we can use the equation which relates $|\vec{k}|$ and ω:

$$|\vec{k}| = \omega/u_\gamma = \omega/(c/n) = n\omega/c. \tag{16}$$

This allows us to write

$$\vec{B}_i + \vec{B}_r = \vec{B}_t \rightarrow \hat{z} \times n_i(\vec{E}_i + \vec{E}_r) = \hat{z} \times n_t \vec{E}_t. \tag{17}$$

We can now solve for the magnitude of the reflected wave in terms of the incident wave:

$$\vec{E}_r = \frac{n_t - n_i}{n_i + n_t}\vec{E}_i. \tag{18}$$

You would have also found this result if you had just used the Fresnel equations with normal incidence ($\theta_i = \theta_t = 0$) and that would have been a correct solution. Now, we are told that $n_i = n$ and $n_t = n(1 + i\rho)$:

$$\frac{n_t - n_i}{n_i + n_t} = \frac{n(1+i\rho) - n}{n + n(1+i\rho)} = \frac{i\rho}{2+i\rho} = \frac{i\rho(2-i\rho)}{4+i\rho} = \frac{\rho^2 + i2\rho}{4+i\rho} = Ae^{i\arctan(2/\rho)}, \tag{19}$$

where A is some real, positive, number (which can be written in terms of n and ρ but which will not affect the phase of the wave). We now note that the waves we are working with can be written as exponentials: $E_i = E_{i,0}e^{i(\vec{k}\vec{x}-\omega t+\varphi)}$ and given that

$$E_r = \frac{n_t - n_i}{n_i + n_t}E_i = Ae^{i\arctan(2/\rho)}E_i \tag{20}$$

we can conclude that the phase difference between the two waves is $\Delta\varphi = \arctan(2/\rho)$.

b. This index of refraction applies to a conductor. For a good conductor $\rho \gg 1$- in this limit what is the phase change?

Solution

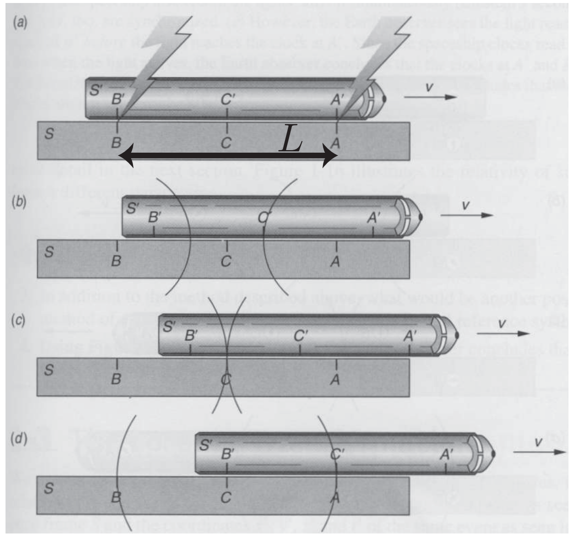
In this case $\Delta\varphi \rightarrow 0$.

3.

In this problem we'll consider the question of simultaneity as described according to Galileo and Einstein. We imagine the same thought experiment that Einstein did in order to explain simultaneity in his popular book on special relativity. The below figure shows two lightening flashes hitting at points A and B and the light then traveling to two observers: one is on the center of the platform and the other on the center of a moving train. As the figure shows, the flashes arrive to the platform observer at the same time, but at different times to the train observer. We will analyze this situation using both Galilean and relativistic perspectives.

a. Draw a space-time diagram describing the situation as seen by an observer on the platform according to a Galilean transformation,

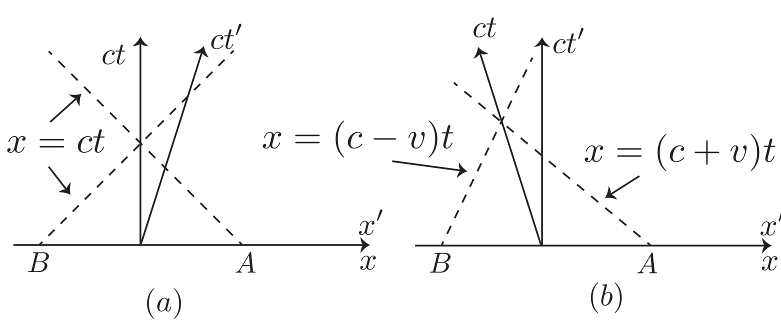
$$x' = x - vt, \tag{21}$$
$$t' = t. \tag{22}$$



Include both the space-time axes for the observer on the platform and the observer on the train and note the slope of the world-lines of the observers and the light from A and B. Assume that according to the person on the platform the speed of light is c. Calculate the the difference in time between when the flashes from flash A and B reach the observer on the train, as measured by the observer on the platform.

b. Draw a space-time diagram describing the situation as seen by an observer on the *train* according to a Galilean transformation. Again, include both the space-time axes for the observer on the platform and the observer on the train and note the slope of the world-lines of the observers and the light from A and B. According to the observer on the *train*, using a Galilean transformation, what is the difference in time between the flash from A and B that they measure? If the observers compare notes, would they agree that the lightening strikes were simultaneous? (This can be answered by referring to your space-time diagram).

Solution



The space-time diagrams for (a) and (b) are shown above. Using normal dynamics for objects moving with a constant velocity, according to the observer on the platform it the two flashes will be seen by the train observer with a time difference,

$$\Delta T_{AB} = \frac{L}{2(c-v)} - \frac{L}{2(c+v)}. \tag{23}$$

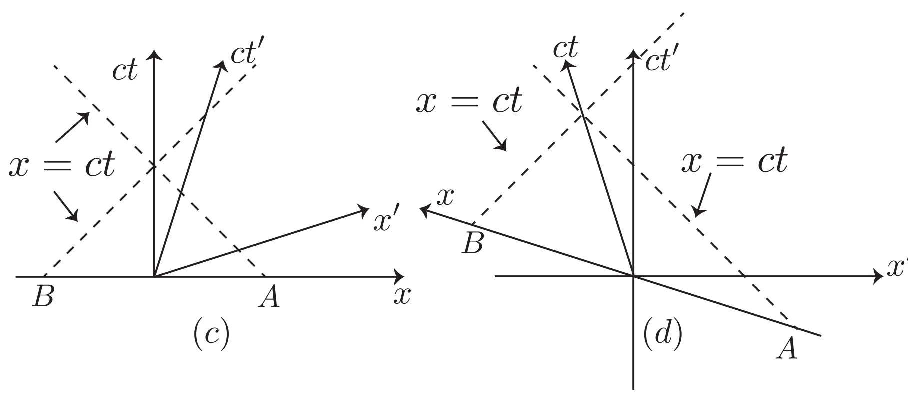
This means that the speed of light, as measured by the train observer, will be $c - v$ for the flash from B and $c + v$ for the flash from A, as shown in the space-time diagram from the train observer's perspective (b). This is because, under a Galilean transformation, the speed of light is different for different inertial observers. However, both the platform observer and the train observer will measure the same time difference, ΔT_{AB}. Both will conclude that the flashes occurred simultaneously, which can be seen in the space-time diagrams because the lines of simultaneity are parallel to the x and x' axes, which, for a Galilean transformation, are the same for both observers.

c. Do the same as in part (a) but using a Lorentz transformation,

$$x' = \gamma(x - \beta ct), \tag{24}$$
$$ct' = \gamma(ct - \beta x). \tag{25}$$

d. Do the same as in part (b) but using a Lorentz transformation.

Solution



For a Lorentz transformation, both observers will measure the speed of light to be the same. The space-time diagram for the platform observer is *almost* identical to the one for part (a), except for the fact that for a Lorentz transformation the x' axis is not identical to the x axis. The difference in the arrival times for the light from flashes A and B are identical for parts (a) and (c).

However, for part (d) things are a bit different. Let's go step-by-step. In the platform frame the flash from A and B reach the train observer at the events

$$\begin{aligned} \mathbf{x}_A &= (cL/(2[c+v]), vL/(2[c+v])), \quad (26) \\ &= \frac{cL}{2(c+v)}(1,\beta), \\ \mathbf{x}_B &= (cL/(2[c-v]), vL/(2[c-v])), \quad (27) \\ &= \frac{cL}{2(c-v)}(1,\beta). \end{aligned}$$

Given the Lorentz transformation

$$x' = \gamma(x - \beta ct), \quad (28)$$
$$ct' = \gamma(ct - \beta x), \quad (29)$$

the same events, in terms of the time and position as measured in S':

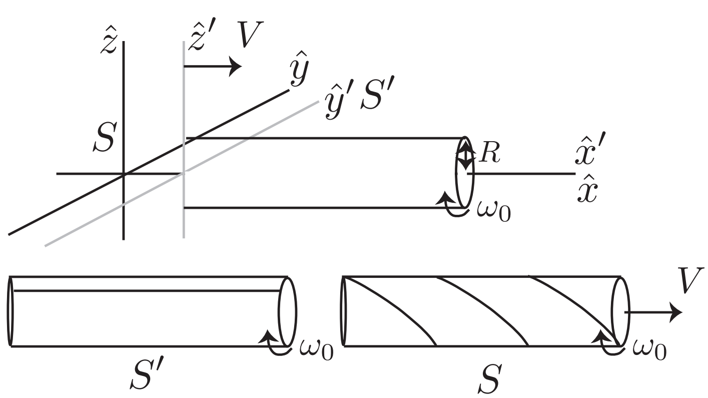
$$\mathbf{x}_A = \gamma L/(2[c+v])\{c - \beta v, v - \beta c\} = \gamma cL/(2[c+v])\{1 - v^2/c^2, 0\} \quad (30)$$
$$= \frac{cL}{2\gamma(c+v)}\{1,0\} \quad (31)$$
$$\mathbf{x}_B = \frac{cL}{2\gamma(c-v)}\{1,0\} \quad (32)$$

This is just a long-winded way to say that time dilation tells us $\Delta t = \gamma\Delta t'$, so that in the rest-frame of the train observer, she measures a time difference of

$$\Delta T'_{AB} = \Delta T_{AB}/\gamma. \quad (33)$$

Also, and most importantly, the two will not agree on whether the flashes occurred simultaneously. Lines of simultaneity are parallel to the x or x' axes, showing that from the perspective of the platform observer the flashes occurred simultaneously, but from the perspective of the train observer, flash A occurs *before* flash B.

4.



It was pointed out by M. von Laue (who did pioneering work in X-ray diffraction) that a cylinder (of radius R) rotating uniformly about the x'-axis of S' will seem *twisted* when observed instantaneously in S, where it not only rotates but also travels forward. See the bottom part of the above figure- in S' a line drawn along the length of the cylinder will just rotate along with the rotation of the cylinder- in S the same line will appear twisted, like a candy cane. If the angular speed of the cylinder in S' is ω_0, show that in S the twist per unit length is $\gamma\omega_0 v/c^2$.

Solution

You can think of the rotating cylinder in frame S' as a bunch of clocks arranged on the x'-axis. Then, because of the Lorentz transformation, the clocks will run at different rates.

If we pick a point $x' = x'_0$ on the cylinder then, in S' it runs at a rate ω_0. This means that the point moves according to:

$$\vec{r}\,'(t') = [0, R\cos(\omega_0 t'), R\sin(\omega_0 t')]. \tag{34}$$

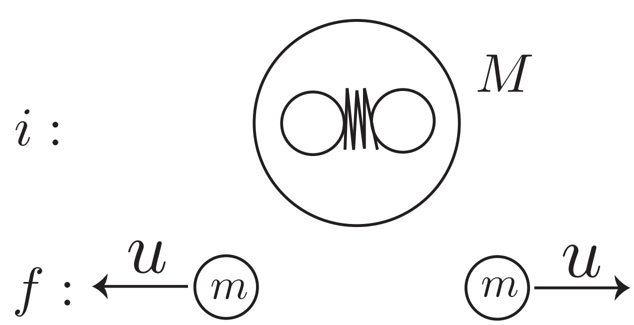
Now in S we use a Lorentz transformation to find that in S, $t' = \gamma(t - v/c^2 x)$. Therefore, that same point moves according to

$$\vec{r}(t) = [0, R\cos[\omega_0\gamma(t - v/c^2 x)], R\sin[\omega_0\gamma(t - v/c^2 x)]]. \tag{35}$$

This shows that not only will each part of the cylinder appear to be rotating *faster* (because of the γ-factor), but different parts of the cylinder have different phases, depending on their x location. For a fixed moment in time the phase difference between two points on the cylinder (again, as seen in S) is $\Delta\phi = -\omega_0\gamma v/c^2\Delta x$. Therefore, per unit length the twist is

$$\frac{\Delta\phi}{\Delta x} = \omega_0\gamma v/c^2. \tag{36}$$

5.



Two billiard balls are initially at rest and connected by a spring, so that the total ball-spring system has mass M. The spring is released and the balls fly apart with equal and opposite speeds, u, and each with rest-mass m.

a. Without using a Lorentz transformation, using the conservation of the total 4-momentum, calculate the final rest-mass of each billiard ball, m, in terms of the initial rest-mass M.

Solution

This problem is trivial if you write down the simplest formulae which describes the dynamics. See the solution in Tipler and Llwellyn for a more complicated derivation (if you do think about why we were able to solve it so quickly but T&L takes so much more time...).

The initial total 4-momentum is

$$\mathbf{p}_{\rm tot,i} = (Mc^2, \vec{0}). \tag{37}$$

The final total 4-momentum is

$$\mathbf{p}_{\rm tot,f} = (2mc^2\gamma(u), \vec{0}). \tag{38}$$

Therefore, by the conservation of the total 4-momentum, the total relativistic energy is conserved, leading to

$$2mc^2\gamma(u) = Mc^2 \to m = \frac{M}{2\gamma(u)}. \tag{39}$$

b. Show that the change in the rest-mass energy, $\Delta mc^2 \equiv (2m - M)c^2$, is equal to -ΔKE.

Solution

Using the result from part (a) we have

$$\Delta mc^2 = (2m - M)c^2 = (2m - 2m\gamma(u))c^2 = 2mc^2(1 - \gamma(u)). \tag{40}$$

We also know that

$$\Delta KE = KE_f - KE_i = 2mc^2(\gamma(u) - 1) - 0 = 2mc^2(\gamma(u) - 1). \tag{41}$$

And we can see that $\Delta mc^2 = -\Delta KE$.