Problem 1. (a) Evaluate $\iint\limits_R y\sin x\ dA$, where R is the plane region bounded by the curve $y = \sin x$ and the lines $y = 0$, $x = 0$ and $x = \pi$.

(b) Let W be the region $\{(x, y, z) \mid \sqrt{x^2 + y^2} \le z \le 1\}$ in space. Write $\iiint\limits_W f\ dV$ as an iterated integral in the order $dz\ dy\ dx$.

Solution (a) The answer was 0. Most of you chose to do it in the $dy\ dx$ order, which does seem to be the more convenient one. One thing you had to watch out for is that when you're setting bounds for y in terms of x, things change drastically as you move from $x < \frac{\pi}{2}$ to $x > \frac{\pi}{2}$ (as will become clear once you draw the region): in the former case $0 < \cos y$, whereas in the latter the inequality is reversed. So you had to break up your integral in two, as in

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos x} y \sin x\ dydx + \int_{\frac{\pi}{2}}^{\pi} \int_{\cos x}^{0} y \sin x\ dydx.$$

Calculating the two inner integrals (that is, integrating with respect to y in both summands), you get

$$(*)\quad \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 x\ \sin x\ dx - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \cos^2 x\ \sin x\ dx.$$

I now claim that the two integrals cancel out. Now, I could just calculate them and see that that's the case, but let's do it more cleverly.

Making the substitution $x = \pi - u$ in the first integral (to turn the interval $\left[0, \frac{\pi}{2}\right]$ into $\left[\frac{\pi}{2}, \pi\right]$), I'm getting

$$\int_0^{\frac{\pi}{2}} \cos^2 x\ \sin x\ dx = \int_{\pi}^{\frac{\pi}{2}} \cos^2(\pi - u)\sin(\pi - u)\ d(\pi - u) = \int_{\frac{\pi}{2}}^{\pi} \cos^2(\pi - u)\sin(\pi - u)\ du.$$

Finally, using $\cos(\pi - u) = -\cos u$ and $\sin(\pi - u) = \sin u$, this last integral is exactly

$$\int_{\frac{\pi}{2}}^{\pi} \cos^2 u \sin u\ du = \int_{\frac{\pi}{2}}^{\pi} \cos^2 x \sin x\ dx.$$

So as claimed, the two integrals in $(*)$ cancel out and you get zero.

(b) The region sits above the cone $z = \sqrt{x^2 + y^2}$ and below the horizontal plane $z = 1$. This already tells you the bounds for z: the lower one's $\sqrt{x^2 + y^2}$, while the upper one is just 1. To find the bounds for x and y, I have to take this solid and project it on the xy plane. Drawing the portion of the solid cone described in the first sentence, you'll see that this projection is nothing but the disk of radius 1 centered at the origin of the xy plane. All in all:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f\ dzdydx.$$

Some of you wrote things like

$$4\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f\ dzdydx$$

instead, presumably trying to exploit the symmetry of the situation. That's incorrect: the region of integration is symmetric, *but the integrand f need not be*! Here, f is a totally arbitrary function (perhaps continuous, but that's all), so you don't know that it has any symmetry properties at all.

2.Let E be the solid bounded by the paraboloid $z = 2(x^2 + y^2)$ and the plane $z = 8$. Find the centroid of E.

Solution.

Using cylindrical coordinates, the region is $\{0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2,\ 2r^2 \leq z \leq 8\}$. Let the density be $\rho = k$,then

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^2\int_{2r^2}^8 krdzdrd\theta \\ &= \int_0^{2\pi}\int_0^2(8-2r^2)krdrd\theta \\ &= 2\pi\int_0^2(8-2r^2)krdr \\ &= 2\pi k(4r^2 - \tfrac{1}{2}r^4)|_0^2 \\ &= 16k\pi \end{aligned}$$

$$\begin{aligned} \bar{z} &= \frac{1}{m}\int_0^{2\pi}\int_0^2\int_{2r^2}^8 zkrdzdrd\theta \\ &= \frac{1}{m}\int_0^{2\pi}\int_0^2 \tfrac{1}{2}z^2|_{2r^2}^8 krdrd\theta \\ &= \frac{1}{m}2\pi\int_0^2(32-2r^4)krdr \\ &= \frac{2k\pi}{m}\int_0^2(32r-2r^5)dr \\ &= \frac{2k\pi}{m}(16r^2 - \frac{r^6}{3})|_0^2 \\ &= \frac{2k\pi}{m}\frac{128}{3} \\ &= \frac{16}{3} \end{aligned}$$

By symmetry, $\bar{x} = \bar{y} = 0$.

Thus, the centroid of E is $(0, 0, \frac{16}{3})$.

3. Let E be the "ice-cream cone" region bounded by the sphere $\rho = 2\cos\phi$ and the cone $\phi = \frac{\pi}{6}$. (We are using spherical coordinates here.) Suppose E is filled with matter with density $d(x, y, z) = z$ (in Cartesian coordinates). Find the mass of E.

Solution

The mass is given by $m = \int\int\int_E d(x,y,z)\ dV$. Converting to spherical, we have $z = \rho\cos\phi$ and pick up the Jacobian $\rho^2 \sin\phi$, so we get the integral

$$m = \int\int\int \rho^3 \cos\phi \sin\phi\ d\rho d\phi d\theta$$

Now, to figure out the bounds, observe that E is given by $\{(\rho, \phi, \theta) | 0 \leq \rho \leq 2\cos\phi, 0 \leq \phi \leq \frac{\pi}{6}, 0 \leq \theta\ 2\pi\}$, so our integral becomes

$$m = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{2\cos\phi} \rho^3 \cos\phi \sin\phi\ d\rho d\phi d\theta$$

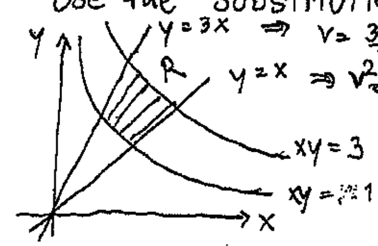
Solving, we get

$$\begin{aligned}
m &= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{2\cos\phi} \rho^3 \cos\phi \sin\phi\ d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \frac{\rho^4}{4} \cos\phi \sin\phi |_0^{2\cos\phi}\ d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \frac{16\cos^4\phi}{4} \cos\phi \sin\phi\ d\phi d\theta \\
&= 4 \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \cos^5\phi \sin\phi\ d\phi d\theta
\end{aligned}$$

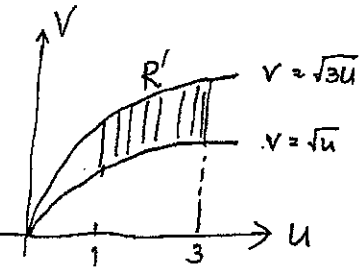
We use the u-substition $u = \cos\phi$, $du = -\sin\phi$ and $\cos 0 = 1$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ to rewrite our integral as

$$\begin{aligned}
m &= 4 \int_0^{2\pi} \int_1^{\frac{\sqrt{3}}{2}} -u^5\ du d\theta \\
&= 4 \int_0^{2\pi} \frac{-u^6}{6} |_1^{\frac{\sqrt{3}}{2}}\ d\theta \\
&= \frac{4}{6} \int_0^{2\pi} [-(\frac{\sqrt{3}}{2})^6 + 1^6]\ d\theta \\
&= \frac{2}{3} \int_0^{2\pi} 1 - \frac{27}{64}\ d\theta \\
&= \frac{2}{3} \cdot \frac{37}{64} \int_0^{2\pi} d\theta = \frac{2}{3} \cdot \frac{37}{64} \cdot 2\pi = \frac{37\pi}{48}
\end{aligned}$$

4. (a) Use the substitution $x = \frac{u}{v}$, $y = v$.



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$



Then

$$\iint_R xy^3\,dA = \iint_{R'} \frac{u}{v}\cdot v^3 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{(u,v)}$$

$$= \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} uv^2\cdot\frac{1}{v}\,dv\,du$$

$$= \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} uv\,dv\,du$$

$$= \int_1^3 \frac{uv^2}{2}\Big|_{v=\sqrt{u}}^{v=\sqrt{3u}} du$$

$$= \int_1^3 u\left(\frac{3u-u}{2}\right)du = \int_1^3 u^2\,du = \frac{u^3}{3}\Big|_{u=1}^{u=3} = \boxed{\frac{26}{3}}.$$

(b) Use the substitution $x = au$, $y = bv$, $z = cw$. $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$

Then $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$ is transformed into $u^2+v^2+w^2 \le 1$ in uvw-coordinates.

∴ Volume of the ellipsoid $= \iiint_{\text{ellipsoid}} 1\,dV(x,y,z)$

$= \iiint_{\substack{\text{sphere}\\\text{of radius } 1}} abc\,dV(u,v,w) = abc \times$ Volume of the sphere

$= \boxed{\frac{4\pi}{3}abc}$.

1. (a) A metal arch has the shape of the curve $y^2 + z^2 = 1, z \geq 0$ in the yz plane. Its density is given by $\rho(x, y, z) = 2 - z$. Compute the mass of the arch.

Solution. First parametrize the curve: $y = \cos t, z = \sin t$ for $0 \leq t \leq \pi$. Using $ds = \sqrt{x'(t)^2 + y'(t)^2}dt = 1 \cdot dt$, we have

$$\begin{aligned} Mass &= \int_C \rho(x, y, z) ds \\ &= \int_0^\pi (2 - \sin t) dt \\ &= [2t + \cos t]_0^\pi = 2\pi - 2 \end{aligned}$$

Parametrization is 2 points worth, Setting up the integral is 2 points worth and correct answer is 1 point worth. We gave 0 point for double, triple integral.

2. (b) Let $\vec{F}(x, y, z) = \langle x^2, xz, y \rangle$. Let C be the line segment from $(0, 0, 0)$ to $(0, 1, 1)$, and let D be the part of the parabola $z = x^2 + 1, y = 1$, from $(0, 1, 1)$ to $(1, 1, 2)$. Let E be the curve C-then-D. Calculate $\int_E \vec{F} \cdot \vec{dr}$.

Solution. Parametrize C: $x = 0, y = t, z = t$ for $0 \leq t \leq 1$. Then $\vec{dr} = \langle x'(t), y'(t), z'(t) \rangle dt = \langle 0, 1, 1 \rangle dt$. Also $\vec{F}$ on C is $\vec{F}(0, t, t) = \langle 0, 0, t \rangle$.

$$\begin{aligned} \int_C \vec{F} \cdot \vec{dr} &= \int_0^1 \langle 0, 0, t \rangle \cdot \langle 0, 1, 1 \rangle dt \\ &= \int_0^1 t dt = \frac{1}{2} \end{aligned}$$

Parametrize D: $x = t, y = 1, z = t^2 + 1$ for $0 \leq t \leq 1$. Then $\vec{dr} = \langle x'(t), y'(t), z'(t) \rangle dt = \langle 1, 0, 2t \rangle dt$. Also $\vec{F}$ on D is $\vec{F}(t, 1, t^2 + 1) = \langle t^2, t(t^2 + 1), 1 \rangle$.

$$\begin{aligned} \int_D \vec{F} \cdot \vec{dr} &= \int_0^1 \langle t^2, t(t^2 + 1), 1 \rangle \cdot \langle 1, 0, 2t \rangle dt \\ &= \int_0^1 t^2 + 2t dt = \frac{4}{3} \end{aligned}$$

So the final answer is $\int_E \vec{F} \cdot \vec{dr} = \int_C \vec{F} \cdot \vec{dr} + \int_D \vec{F} \cdot \vec{dr} = \frac{1}{2} + \frac{4}{3} = \frac{11}{6}$

Problem 6(a) *Find a constant a such that the vector field $\mathbf{F}(x,y) = \langle ax^2y - y^3, 3x^2 - 3xy^2\rangle$ is conservative, or else show that there is no such constant a.*

If a is constant then

$$\frac{\partial P}{\partial y} = ax^2 - 3y^2 \quad \text{and} \quad \frac{\partial Q}{\partial x} = 6x - 3y^2.$$

If $\mathbf{F}$ is conservative then $\partial P/\partial y = \partial Q/\partial x$, but no constant a can make these two things equal. Hence there is no such constant a making the vector field conservative.

Problem 6(b) *Let $\mathbf{F}(x,y) = \langle 6xy - y^3, 3x^2 - 3xy^2 + 4y\rangle$. Find a scalar potential for $\mathbf{F}$.*

We want a function f such that

$$f_x(x,y) = 6xy - y^3 \quad \text{and} \quad f_y(x,y) = 3x^2 - 3xy^2 + 4y.$$

Let

$$f = \int f_x(x,y)\,dx = 3x^2y - xy^3 + g(y).$$

Then

$$f_y(x,y) = 3x^2 - 3xy^2 + g'(y), \quad \text{so} \quad g'(y) = 4y, \quad \text{so} \quad g(y) = 2y^2 + C.$$

Therefore

$$f(x,y) = \boxed{3x^2y - xy^3 + 2y^2 + C}.$$

Problem 6(c) *For $\mathbf{F}$ the field from part (b), find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $C = \{(\cos t, \sin t) : t \in [0,\pi]\}$, oriented from $(1,0)$ to $(-1,0)$.*

By the Fundamental Theorem for Line Integrals, using the scalar potential $f(x,y) = 3x^2y - xy^3 + 2y^2$:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(-1,0) - f(1,0) = 0 - 0 = \boxed{0}.$$

7. True or False. (There is no penalty for guessing wrong. In the questions below, a *nice* scalar or vector field is one that has continuous partial derivatives of all orders.

F 1. Let f be a nice scalar field on $\mathbb{R}^2$; then for any smooth oriented curve C in $\mathbb{R}^2$, letting $-C$ be C with the opposite orientation, we have $\int_C f\,ds = -\int_{-C} f\,ds$.

Line integral of a scalar field does not depend on orientation.

T 2. If f is continuous on $[0,1]$, then $\int_0^1 \int_0^1 f(x)f(y)\,dy\,dx = (\int_0^1 f(x)\,dx)^2$.

Comes from Fubini theorem.

F 3. $\int_0^1 \int_0^{x^2} f(x,y)\,dy\,dx = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$ for any continuous f on $\mathbb{R}^2$.

Not the same region on both sides.

T 4. If f is continuous on $\mathbb{R}^2$, and R_a is the square with edge-length a centered at (x_0, y_0), then

$$f(x_0, y_0) = \lim_{a \to 0} \left(\frac{1}{a^2} \iint_{R_a} f\,dA \right).$$

Value of function = limit of average values on neighborhoods around the point.

T 5. Let S_1 and S_2 be type I regions in the uv-plane, and suppose they are mapped to regions R_1 and R_2 in the xy-plane, respectively, by the transformation $T(u,v) = (2u+v, 3v)$. Then $\dfrac{\text{area}(R_1)}{\text{area}(S_1)} = \dfrac{\text{area}(R_2)}{\text{area}(S_2)}$.

Jacobian is constant.

T 6. Let C be a smooth curve in $\mathbb{R}^3$, and suppose g is a nice scalar field such that $g(x,y,z) = k$ for all points (x,y,z) on C. Then $\int_C \nabla g \cdot d\vec{r} = 0$.

If A = initial pt and B = final point, $\int_C \nabla g \cdot d\vec{r} = g(B) - g(A) = 0$.

T 7. The vector field $\vec{F}(\vec{r}) = \dfrac{\vec{r}}{|\vec{r}|^3}$ is independent of path on $\mathbb{R}^2 - \{(0,0)\}$.

The scalar field $f(\vec{r}) = \dfrac{-1}{|\vec{r}|}$ is a potential.

F 8. Let $\vec{F} = \langle P, Q, R \rangle$ and $\vec{G} = \langle P, S, T \rangle$ be continuous vector fields with the same first coordinate function $P(x,y,z)$. Let C be an oriented smooth curve lying in the plane $z = 4$. Then $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}$.

No, you'd need $Q = S$ as well. E.g. $\vec{F} = \langle 0, 1, 0 \rangle$, $\vec{G} = \langle 0, 2, 0 \rangle$, C = line from $(0,0,4)$ to $(0,1,4)$.