Math 53, First Midterm

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Discussion section: ∞

Instructions: *Please show your work: unjustified answers will not receive credit. Use back of page if needed. (No justification is required in the True/False section, however.) Your signature above certifies that the work here is your own.*

1. (a) Define carefully: f is differentiable at (a, b) if and only if... letting

$\Delta x = x - a$, $\Delta y = y - b$, and

$\Delta z = f(x,y) - f(a,b)$, we have

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

(b) Given an example of a function f which is continuous at $(0, 0)$, but not differentiable at $(0, 0)$. (No justification needed.)

Some examples are

$$f(x,y) = |x|$$

and

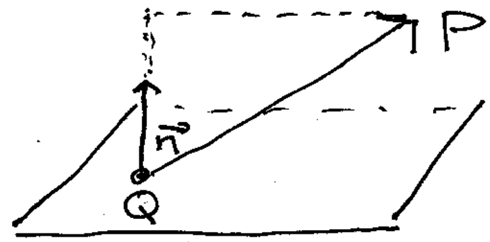
$$f(x,y) = \sqrt{x^2+y^2}$$

(c) What hypotheses on f let you conclude (via a major theorem in the book) that f is differentiable at (a, b)?

If f_x and f_y are continuous at (a,b), and defined on a disk around (a,b), then f is differentiable at (a,b).

2. (a) Find the distance from the point $P(4, -1, 2)$ to the plane $2x + 2y - z + 1 = 0$.

$\vec{n} = \langle 2, 2, -1 \rangle$ is normal to the plane. Let $Q = (0, 0, 1)$, so that Q is on the plane. Set $\vec{b} = \overrightarrow{QP} = \langle 4, -1, 1 \rangle$. The distance is



$$|\operatorname{comp}_{\vec{n}}(\vec{b})| = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{n}|} \right|$$

$$= \left| \frac{\langle 4, -1, 1 \rangle \cdot \langle 2, 2, -1 \rangle}{\sqrt{9}} \right|$$

$$= \left| \frac{8 - 2 - 1}{3} \right| = \frac{5}{3}$$

(b) Determine whether the points $(1, 0, 1)$, $(4, 1, 3)$, $(2, 2, 2)$, and $(3, 0, 3)$ lie in the same plane.

$P \quad Q \quad R \quad S$

$\overrightarrow{PQ} = \langle 3, 1, 2 \rangle$, $\overrightarrow{PS} = \langle 2, 0, 2 \rangle$

$\overrightarrow{PR} = \langle 1, 2, 1 \rangle$. The points are coplanar just in case the parallelipiped they determine has zero volume. The volume is the absolute value of the scalar triple product:

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 3(4) - 1 \cdot (2 - 2) + 2(-2)$$

$$= 8 \neq 0$$

Since the volume is nonzero, the points are <u>NOT</u> coplanar.

3. (a) Find an equation for the plane tangent to the surface $xyz^2 = 8$ at the point $(1, 2, 2)$.

Let $f(x,y,z) = xyz^2$. Then $\nabla f(1,2,2)$ is normal to the plane in question. But

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz^2, xz^2, 2xyz \rangle$$

$$\nabla f(1,2,2) = \langle 8, 4, 8 \rangle$$

The point-normal equation for the plane is then

$$\langle 8,4,8 \rangle \cdot (\langle x,y,z \rangle - \langle 1,2,2 \rangle) = 0$$

or

$$8(x-1) + 4(y-1) + 8(z-2) = 0$$

(b) Is there a point on the surface $xyz^2 = 8$ where the tangent plane is parallel to the xy plane? (Justify.)

At such a point (x,y,z), we must have $\nabla f(x,y,z)$ is a normal to the xy plane, and hence a nonzero scalar multiple of $\langle 0,0,1 \rangle$. So we want to know whether there are x,y,z and $c \neq 0$ such that

$$\langle yz^2, xz^2, 2xyz \rangle = c\langle 0,0,1 \rangle$$

and

(x,y,z) is on the surface.

This implies $yz^2 = 0$, $xz^2 = 0$, $2xyz = c$ and $xyz^2 = 8$.

But $yz^2 = 0$ implies $y = 0$ or $z = 0$, and in either case, $xyz^2 = 0 \neq 8$. So there is no such point.

4. Locate all local maxima, local minima, and saddle points for the function $f(x,y) = x^2 + y^2 + x^2y + 4$.

$f_x = 2x + 2xy = 0$ implies $x + xy = 0$

$x(1+y) = 0$

$x = 0$ or $y = -1$

$f_y = 2y + x^2 = 0$ implies $x^2 = -2y$.

So if $x = 0$, $y = 0$

if $y = -1$, $x^2 = 2$, $x = \pm\sqrt{2}$

So three critical points, $(0,0)$, $(\sqrt{2}, -1)$, $(-\sqrt{2}, -1)$.

$f_{xx} = 2 + 2y \qquad f_{xy} = 2x$

$f_{yy} = 2$

At $(0,0)$: $f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - 0 = 4 > 0$ not saddle

since $f_{yy} = 2 > 0$, local min.

At $(\sqrt{2}, -1)$: $f_{xx} f_{yy} - f_{xy}^2 = (2-2)2 - (2\sqrt{2})^2 = -8 < 0$

Saddle point

At $(-\sqrt{2}, -1)$: $f_{xx} f_{yy} - f_{xy}^2 = (2-2)2 - (-2\sqrt{2})^2 = -8 < 0$

Saddle point

5. Let $z = f(x, y)$, for $x = x(t) = t^2$ and $y = y(t) = t^3$. Suppose $f_x(1,1) = 2$, $f_y(1,1) = 3$, $f_{xx}(1,1) = 1$, $f_{yy}(1,1) = 2$, and $f_{xy}(1,1) = f_{yx}(1,1) = -1$.

(a) Compute $\dfrac{dz}{dt}$, when $t = 1$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

using $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$

So when $t=1$

$$\frac{dz}{dt} = 2\cdot 2 + 3\cdot 3 = 13$$

(b) Compute $\dfrac{d^2z}{dt^2}$ when $t = 1$.

$$\frac{d^2z}{dt} = \frac{d}{dt}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial z}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial z}{\partial y}\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}$$

But

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y\partial x}\frac{dy}{dt}\right) = 1\cdot 2t + (-1)3t^2$$

$$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right) = (-1)2t + 2\cdot 3t^2$$

when $t=1$

$$\frac{d^2z}{dt^2} = (2-3)(2) + 2\cdot 2 + (6-2)3 + 3\cdot 6$$

$$= -2 + 4 + 12 + 18 = 32$$

6. (a) A particle moves in 3-space in such a way that its position vector is always orthogonal to its velocity vector. Show that the particle is moving on the surface of a sphere centered at the origin.

Let $\vec{r}(t)$ = position of particle at time t.

We must show that $|\vec{r}(t)|$ is constant in time.

But $\frac{d}{dt}\left(|\vec{r}|^2\right) = \frac{d}{dt}\left(\vec{r}\cdot\vec{r}\right) = 2\vec{r}\cdot\frac{d\vec{r}}{dt}$

Our hypothesis is that $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0$ for all t.

Thus $\frac{d}{dt}\left(|\vec{r}(t)|^2\right) = 0$ ~~is [illegible]~~ so $|\vec{r}|^2$ is constant, so $|\vec{r}(t)|$ is constant.

(b) Find the rate of change of the function $f(x, y, z) = 5x^2 - 3xy + xyz$ at the point $P(3, 4, 5)$, in the direction of the point $Q(4, 5, 4)$.

$$\vec{PQ} = \langle 1, 1, -1 \rangle$$

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{1}{\sqrt{3}}\langle 1, 1, -1 \rangle$$

$$\nabla f = \langle 10x - 3y + yz, -3x + xz, xy \rangle$$

$$\nabla f(3,4,5) = \langle 38, 6, 12 \rangle$$

$$D_{\vec{u}}(f)(3,4,5) = \nabla f(3,4,5)\cdot\vec{u}$$

$$= \langle 38, 6, 12 \rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, -1 \rangle$$

$$= \frac{1}{\sqrt{3}}(38 + 6 - 12)$$

$$= 32/\sqrt{3}$$

6. **True or False.** (There is no penalty for guessing wrong.)

T 1. For any vectors $\vec{a}$ and $\vec{b}$, if $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$, then $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$.

F 2. If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector-valued functions, then $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)]=\vec{u}'(t)\times\vec{v}'(t)$.

F 3. If $f(x,y)\to L$ as $(x,y)\to(a,b)$ along every straight line, then $\lim_{(x,y)\to(a,b)} f(x,y)=L$.

T 4. If all partials of F exist and are continuous everywhere, and $\nabla F(a,b,c)\neq\vec{0}$, then the equation $F(x,y,z)=F(a,b,c)$ defines a surface near (a,b,c).

F 5. Let $R(x,y)=f(x,y)-(f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b))$. Then f is differentiable at (a,b) if and only if $R(x,y)\to 0$ as $(x,y)\to(a,b)$.

F 6. If $f_x(a,b)$ and $f_y(a,b)$ exist, then f is differentiable at (a,b).

T 7. If f is differentiable at (a,b), and $\nabla f(a,b)\neq\vec{0}$, then the derivative of f at (a,b) in the direction tangent to the curve $f(x,y)=f(a,b)$ is zero.

T 8. If all second partial derivatives of f exist and are continuous everywhere, then $f_{xy}=f_{yx}$. Clairaut's theorem

Explanations:

(1) If neither $\vec{a}$ nor $\vec{b}$ is $\vec{0}$, then for θ the angle between them, $\cos\theta=0$ and $\sin\theta=0$. This is impossible.

(2) $\frac{d}{dt}(\vec{u}\times\vec{v})=(\vec{u}'\times\vec{v})+(\vec{u}\times\vec{v}')$

(3) There are other ways to approach

(4) This is the Implicit Function Theorem

(5) This just said f is continuous at (a,b).

(6) No way! Two tangent lines doesn't give you a tangent plane.

(7) Let $\vec{r}(t)$ parametrize the curve, with $\vec{r}(0)=\langle a,b\rangle$.
Then $\frac{d}{dt}(f(\vec{r}(t)))=0=\nabla f(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}$ at $t=0$.