1.1a Prove e^z is nonzero $\forall z \in \mathbb{C}$

Slick solution Note $e^z e^{-z} = 1 \implies e^z \neq 0$. While this solution is very easy to understand, it's not obvious. Very few students saw this solution.

More common solution $e^z = e^a e^{ib}$. We know from high school calculus that for all real a, $e^a > 0$. From 1.b, we know $|e^{ib}| = 1$ and thus $e^{ib} \neq 0$. Thus $e^z \neq 0$. We initially did not give credit for this because we decided one needed to prove, $e^a > 0$ because the proof is non-trivial, but later decided it was fine. We took off points and you may submit them for regrade.

Less common solution Note that for any log function, $\log(0)$ is undefined (if you said it was $-\infty$, we still gave you points even though this is incorrect). Also note that there is a log function s.t. $\log(e^z) = z$. Therefore it is defined at $\log(e^z)$ and thus e^z cannot be 0.

Correct solutions earned 10 points. If you at some point mentioned $e^z = e^a e^{ib}$ and $e^a > 0$, then you will receive at least 5 points.

1.1b Prove $|e^{i\theta}| = 1 \; \forall \theta \in \mathbb{R}$

The main solution $|e^{i\theta}| = |cos(\theta) + isin(\theta)| = \sqrt{cos^2(\theta) + sin^2(\theta)} = \sqrt{1} = 1$.
The solution received 10 points. Some students said that $|a + ib| = a^2 + b^2$. These students received 9 points.

Alternate solution We know from lecture that $f(t) = e^{it}$ has traces a circle and thus has constant magnitude. Furthermore $|f(0)| = |1|$ and thus $|f(\theta)| = |e^{i\theta}| = 1$. Some students commented on how it traced out the unit circle. However, we could not accept this solution because it's a circular argument. The image of f is a circle with a radius. We call it the unit circle because we can show it has radius 1. However, when showing it has radius 1, we cannot use this fact because it depends on us knowing it has radius 1.

Other solutions Some students said that $|e^{i\theta}| = |re^{i\theta}| = r$. This solution has a very subtle flaw. How do we know that $|re^{i\theta}| = r$? Well, $|re^{i\theta}| = |r||e^{i\theta}| = r|e^{i\theta}|$. So, if $|e^{i\theta}| = 1$, then the above statement holds. But it depends on us showing this.

1.1c This question asks you to prove an if and only if statement. Remember that when you are asked to prove an if and only if statement, you have to prove it both ways, the if direction and the only if direction.

if direction Here you try to prove that if $z = i2\pi k$, then $e^z = 1$.

$$\begin{aligned} z &= i2\pi k \\ e^z &= e^{i2\pi k} = \cos 2\pi k + i \sin 2\pi k \qquad \text{(Euler's Formula)} \end{aligned}$$

Since $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$ for any integer k, this proves that

$$e^z = e^{i2\pi k} = \cos 2\pi k + i \sin 2\pi k = 1 + 0i = 1$$

Notes

- The if proof is worth 5 points (out of the total of 10 points for 1c).
- Partial proof is worth 3 points.
- You cannot state directly that $e^{i2\pi k}$ is just 1. That is exactly what you are asked to prove. Even if you've learned in class and you can see "obviously" that $e^{i2\pi k}$ is indeed 1, you cannot just say that and be done with the proof. That's simply restating the problem!
- To continue the previous bullet point, you can't just draw a Real-Imaginary plane, draw the 1 vector and say that is also $e^{i2\pi k}$ and be done with the problem. Again, you have not shown "why" that is the case and only restated the problem (albeit in a graphical way).

only if direction Here you try to prove that if $e^z = 1$, then $z = i2\pi k$.

$$\begin{aligned} e^z &= 1 \\ e^z &= e^{a+ib} \quad a, b \in \mathbb{R} \\ e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b) &= 1 = 1 + 0i \\ \Rightarrow e^a \cos b &= 1,\ e^a \sin b = 0 \end{aligned}$$

The last step comes the fact the real parts must be equal, and the imaginary parts must be equal.
$e^a > 0$ for any real number a. Therefore:

$$e^a \sin b = 0 \Rightarrow \sin b = 0 \Rightarrow b = n\pi, n \in \mathbb{Z}$$

$$b = n\pi \Rightarrow \cos b = \cos n\pi = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} = (-1)^n$$

So we now have

$$e^a \cos b = e^a(-1)^n = 1$$

Again, $e^a > 0$ for any real number a. Therefore for the above equation to hold, the other term, $(-1)^n$, has to be positive, otherwise the product will be negative and not equal to 1. This means that we must have:

$$\cos b = (-1)^n = 1$$

Which then implies that

$$e^a \cos b = e^a = 1 \Rightarrow a = 0$$

And

$$(-1)^n = 1 \Rightarrow n \text{ is even} \Rightarrow b = n\pi = 2\pi k \quad k \in \mathbb{Z}$$

We've successfully shown that

$$z = a + ib, a = 0, b = 2\pi k$$

Which means

$$z = i2\pi k$$

Notes

- The only if proof is worth 5 points (out of the total of 10 points for 1c).
- Partial proof is worth 3 points. By partial, we mean you have either proved that $a = 0$ or $b = 2\pi k$.
- Note that $\sin b = 0$ only implies that b is a multiple of π, not b is an even multiple of π, because sin is equal to zero for odd multiples of π as well. So you can't get $b = 2\pi k$ just from the sin part.
- Note that $z = i\theta$ is not true! (θ is a real number here). Remember that z is a general complex number, that means it can have real parts. $i\theta$ is a purely imaginary number.

1.1d The statement was *false*. Here's a proof by counterexample. Let

$$z_1 = 0, z_2 = 2\pi i$$
$$e^{z_1} = e^0 = 1 = e^{2\pi i} = e^{z_2}$$
$$z_1 \neq z_2$$

5 points if you stated *false* and provided a correct proof.
4 points if there was a small mistake in the proof.
3 points if you stated *false* but the proof you provided did not apply to the general complex number z
0 points if you stated *true* or if you stated *false* and provided no reasoning

1.2a First Method:

$$
\begin{aligned}
e^{i\theta} + e^{i\phi} &= e^{\frac{i(\theta+\phi)}{2}}(e^{\frac{i(\theta-\phi)}{2}} + e^{\frac{i(\phi-\theta)}{2}}) \quad (4pts) \\
&= e^{\frac{i(\theta+\phi)}{2}}(e^{\frac{i(\theta-\phi)}{2}} + e^{\frac{-i(\theta-\phi)}{2}}) \quad (4pts) \\
&= 2e^{\frac{i(\theta+\phi)}{2}} cos(\frac{\theta-\phi}{2}) \quad (4pts)
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= 2 \quad (1.5pts) \\
\beta &= \tfrac{1}{2} \quad (1.5pts)
\end{aligned}
$$

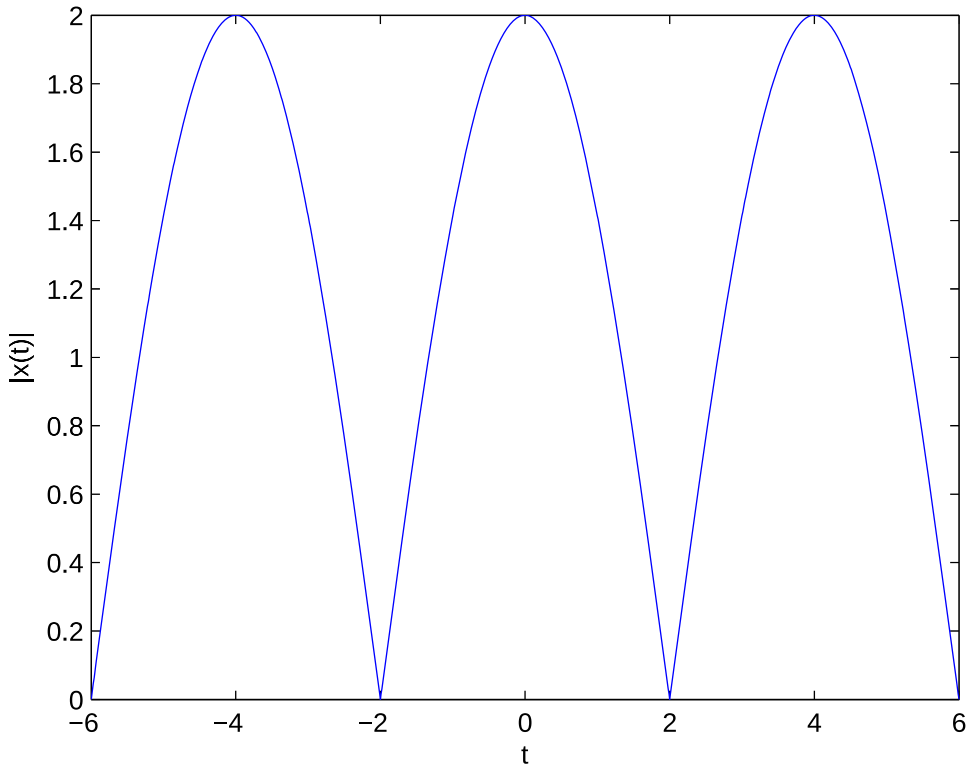
Second Method:

$$
\begin{aligned}
e^{i\theta} + e^{i\phi} &= (cos\theta + cos\phi) + i(sin\theta + sin\phi) \quad (3pts) \\
&= 2cos(\frac{\theta+\phi}{2})cos(\frac{\theta-\phi}{2}) + i2sin(\frac{\theta+\phi}{2})cos(\frac{\theta-\phi}{2}) \quad (3pts) \\
&= 2cos(\frac{\theta-\phi}{2})(cos(\frac{\theta+\phi}{2}) + isin(\frac{\theta+\phi}{2})) \quad (3pts) \\
&= 2cos(\frac{\theta-\phi}{2})e^{\frac{i(\theta+\phi)}{2}} \quad (3pts)
\end{aligned}
$$

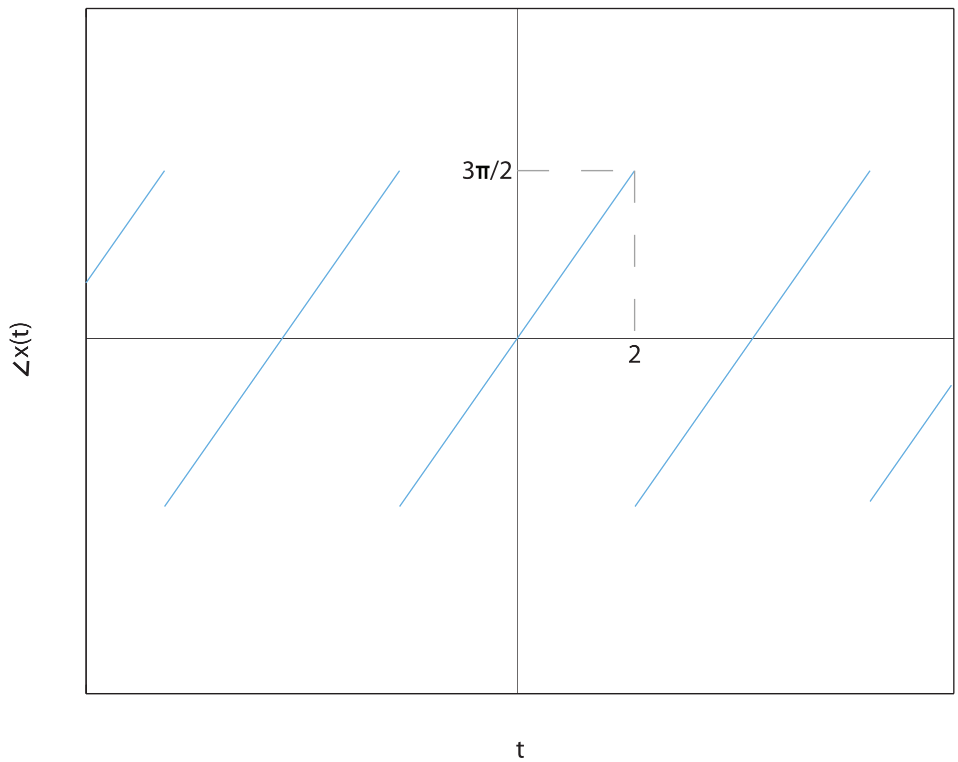
$$
\begin{aligned}
\alpha &= 2 \quad (1.5pts) \\
\beta &= \tfrac{1}{2} \quad (1.5pts)
\end{aligned}
$$

1.2b

- +3 points for almost any work shown to derive Magnitude and Phase.
- Manitude plot (6 pts)
  - Shape (3 pts)
    - +3 if completely correct shape OR
    - +2 if the shape simply resembles a shifted up version of a sine wave. OR
    - +1.5 if the shape is an unshifted sine wave.
  - Labelling (3 pts)
    - +1 Labelling t on the horizontal axis
    - +1 For labelling the correct peak value = $\alpha = 2$
    - +1 For labelling the zero crossing value = $1/\beta = 2$



- Phase plot (6 pts)
  - Shape (3 pts)
    - +3 if completely correct shape OR
    - +1 if some sort of periodicity is shown OR
    - +1 if graphed a linear line
  - Labelling (3 pts)
    - +1 Labelling t on the horizontal axis
    - +1 For labelling the correct peak value $= 3\pi/2$
    - +1 For labelling the zero crossing value $= 1/\beta = 2$



1.3a Using the quadratic formula,

$$z_{A,1}, z_{A,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \frac{\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2} - 1}}{2} = \frac{1}{2}\left(\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i\right) = \frac{1}{2}e^{\pm i\frac{\pi}{4}}$$

$$z_{B,1}, z_{B,2} = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2b_0}}{2b_2} = \frac{\frac{1}{\sqrt{2}} \pm \sqrt{\frac{1}{2} - 1}}{\frac{1}{2}} = 2\left(\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i\right) = 2e^{\pm i\frac{\pi}{4}}$$

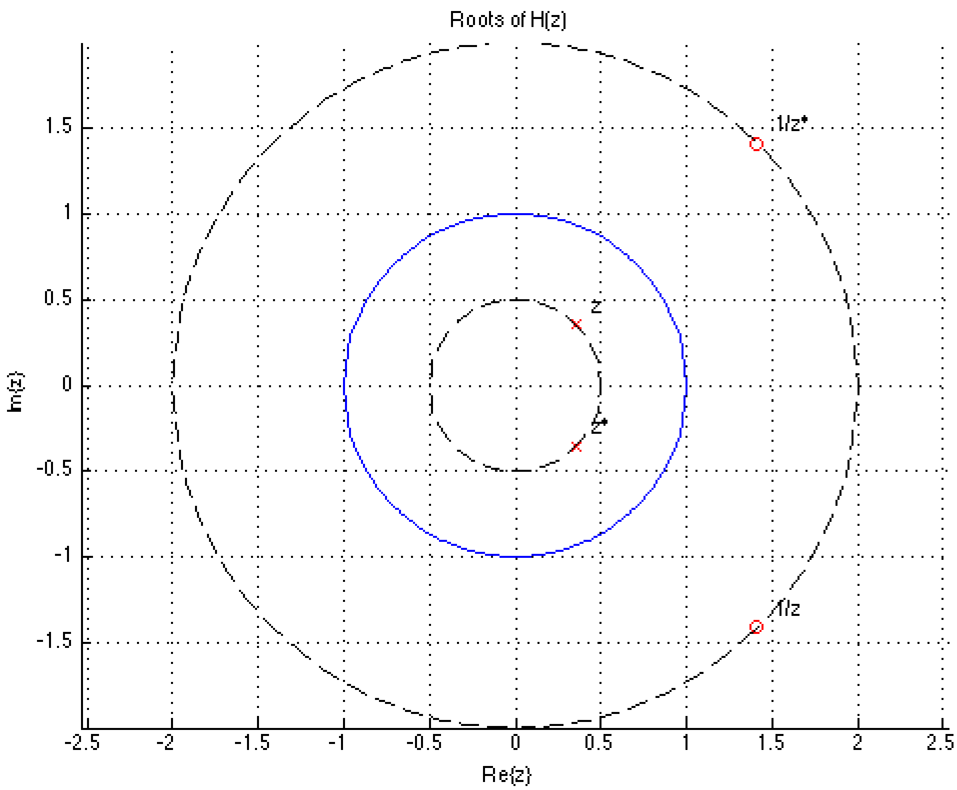
.

When expressed in polar form, it is clear that $z_{A,1} = z_{B,2}^{-1}$ and $z_{A,2} = z_{B,1}^{-1}$.

1. Correct roots for $\hat{A}(z)$ (6 points)
   - Mathematically correct solution (5 points)
   - Simplified and clear answer (1 point)
2. Correct roots for $\hat{B}(z)$ (6 points)

- Mathematically correct solution (5 points)
- Simplified and clear answer (1 point)

3. Relationship of the roots (3 points total)
   - The roots of $\hat{A}(z)$ have magnitudes that are inverses of the magnitudes of the roots of $\hat{B}(z)$ (2 out of 3 points).
   - The roots of $\hat{A}(z)$ are reciprocals (or inverses) of the roots of $\hat{B}(z)$. (3 out of 3 points).



1.3b 1. Graphing roots (4 points total)

- For each graphed pole or zero:
  - Graphed solution from (1.3a) correctly ($\frac{1}{2}$ point)
  - Correct solution of (1.3a) ($\frac{1}{2}$ point
- Incorrect or lacking use of X's and O's (-2 points total for this part)

2. Graph labels (2 points)
   - Labeling axes as Real and Imaginary
   - Labeling unit circle
3. Description (4 points)
   Because this question was very similar to the quiz, the description was weighted more when graded. Full points were given for any mention of the following:
   - The poles and zeros are reciprocals (or inverses) of each other.
   - The poles are symmetric about the real axis, and so are the zeros.

1.3c

$$H(\omega) = \frac{\frac{1}{4}e^{i2\omega} - \frac{1}{\sqrt{2}}e^{i\omega} + 1}{e^{i2\omega} - \frac{1}{\sqrt{2}}e^{i\omega} + \frac{1}{4}} = e^{i2w}\frac{\frac{1}{4} - \frac{1}{\sqrt{2}}e^{-i\omega} + e^{-i2\omega}}{e^{i2\omega} - \frac{1}{\sqrt{2}}e^{i\omega} + \frac{1}{4}}$$

$$\Rightarrow |H(\omega)| = |e^{i2\omega}|\left|\frac{\frac{1}{4} - \frac{1}{\sqrt{2}}e^{-i\omega} + e^{-i2\omega}}{e^{i2\omega} - \frac{1}{\sqrt{2}}e^{i\omega} + \frac{1}{4}}\right| = \frac{|\frac{1}{4} - \frac{1}{\sqrt{2}}e^{-i\omega} + e^{-i2\omega}|}{|e^{i2\omega} - \frac{1}{\sqrt{2}}e^{i\omega} + \frac{1}{4}|}$$

Observe that the denominator of the last equation is the complex conjugate of the numerator, and vice versa, therefore

$$|H(\omega)| = 1$$

Grading rubric: plugging in $z = e^{iw}$ to $H(z)$, 3 points; taking out e^{i2w} or similar tricks, 5 points; conjugate argument, 5 points; final solution, 2 points.