① Prof. Huang Midterm 1, Problem 1, Fall 2007 Solution

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a)

density of atoms $= \frac{N}{V} = 1$ ~~cm^{-3}~~

$= \frac{1}{cm^3}\left(\frac{100\,cm}{1\,m}\right)^3 = 10^6\ /m^3 \equiv n$

by ideal gas law, $\frac{N}{V} = \frac{P}{Tk_B}$

$P = nk_BT = (10^6\,m^{-3})(1.38\,e{-23}\,J/K)(3000K)$

$= 4.14\,e{-14}\,Pa\left(\frac{1\,atm}{1.01e5\,Pa}\right)\left(\frac{760\,torr}{1\,atm}\right)$

$\boxed{P = 3.1\,e{-16}\ torr}$

b) $l_f \equiv \frac{1}{(N/V)(4\pi r^2)\sqrt{2}}$; $d_H = 10^{-10}\,m$; $r_H = \frac{d_H}{2}$

$\rightarrow r_H = 5\,e{-11}\,m$

$$l_f = \frac{1}{(1\,e6\,m^{-3})(5e{-11}m)^2(4\pi)\sqrt{2}}$$

$\boxed{l_f = 2.25\,e\,13\ m}$

a light year is $9.5\,e15\,m$, so the mean free path is $\sim \frac{1}{100}$ of the distance traveled by light in a year.

2. $T_H = 900\,k \quad T_L = 350\,k$

$Q_{in} = 1800\,J \quad W = 700\,J$

$Q_{out} = Q_{in} - W$ (1st law of thermodynamics)

$Q_{out} = 1100\,J$

a. $e = \dfrac{W}{Q_{in}} = \dfrac{700\,J}{1800\,J} = \dfrac{7}{18} = 0.389$

b. $e^c = 1 - \dfrac{T_L}{T_H} = 1 - \dfrac{350\,k}{900\,k} = 1 - \dfrac{7}{18} = \dfrac{11}{18}$

$= 0.611$

c. $\Delta S_{universe} = \Delta S_{Hot\ Reservoir} + \Delta S_{Cold\ Reservoir} + \Delta S_{engine}$

$= \int \dfrac{-dQ_{in}}{T_H} + \int \dfrac{dQ_{out}}{T_L} + 0$

engine returns to original state, and entropy is a state variable

$= -\dfrac{Q_{in}}{T_H} + \dfrac{Q_{out}}{T_L}$

since T is constant for reservoirs

$= -\dfrac{1800\,J}{900\,k} + \dfrac{1100\,J}{350\,k} = 1.14\,\dfrac{J}{K}$

2, cont'd.) d)

Carnot engines are reversible.

$$\Delta S^{c}_{universe} = 0 \ \frac{J}{K}$$

$\Delta S^{c}_{universe}$ is change in entropy for carnot engine.

(alternatively, $Q^{c}_{in} = 1800 J$, $W^{c} = e^{c} Q_{in} = 1100 J$,

$Q^{c}_{out} = 700 J$,

$$\Delta S^{c}_{universe} = -\frac{Q^{c}_{in}}{T_H} + \frac{Q^{c}_{out}}{T_L}$$

$$= -\frac{1800 J}{900 K} + \frac{700 J}{350 K} = 0 \ \frac{J}{K})$$

e)

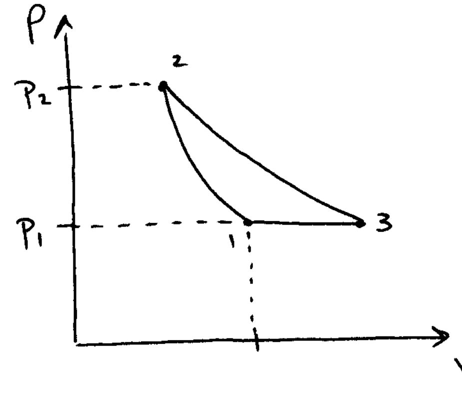
$$\Delta S = \Delta S_{universe} - \Delta S^{c}_{universe}$$

$$= -\frac{Q_{in}}{T_H} + \frac{Q_{out}}{T_L} - 0$$

$$T_L \Delta S = -\frac{T_L}{T_H} Q_{in} + Q_{out}$$

$$= \underbrace{-\frac{T_L}{T_H} Q_{in} + Q_{in}}\, - W$$

$$= Q_{in} \underbrace{\left(1 - \frac{T_L}{T_H}\right)}_{e^{c}} - W = W^{c} - W$$

3. 

n moles of a monoatomic gas $(d=3)$ $\hookrightarrow \gamma = \frac{5}{3}$

$1 \to 2$ adiabatic
$2 \to 3$ isothermal
$3 \to 1$ isobaric

Find Q and W for each process in terms of n, P_1, P_2, V_1

(a) Adiabatic Process $1 \to 2$

$\boxed{Q=0}$

$$\Delta E = -W = \frac{d}{2} nR\,\Delta T = \frac{3}{2} nR\,(T_2 - T_1)$$

$$= \frac{3}{2} nR\left(\frac{P_2 V_1}{nR}\left(\frac{P_1}{P_2}\right)^{1/\gamma} - \frac{P_1 V_1}{nR}\right)$$

$$\boxed{W = \frac{3}{2} V_1\left(P_1 - P_2\left(\frac{P_1}{P_2}\right)^{1/\gamma}\right)}$$

$nRT_1 = P_1 V_1$

$$\boxed{T_1 = \frac{P_1 V_1}{nR}}$$

$nRT_2 = P_2 V_2$

$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$

$V_2 = \left(\frac{P_1}{P_2}\right)^{1/\gamma} \cdot V_1$

$$\longrightarrow \boxed{T_2 = \frac{P_2 V_1}{nR} \cdot \left(\frac{P_1}{P_2}\right)^{1/\gamma}}$$

(b) Isothermal process $2 \to 3$

$\Delta E = \frac{d}{2} NK\,\Delta T \qquad \Delta T = 0 \longrightarrow \Delta E = 0 \qquad \underline{Q = W}$

$$W = \int_{V_2}^{V_3} p\,dV = \int_{V_2}^{V_3} \frac{nRT_2}{V}\,dV = nRT_2 \ln\left(\frac{V_3}{V_2}\right)$$

$$= P_2 V_1 \left(\frac{P_1}{P_2}\right)^{1/\gamma} \ln\left[\frac{P_2}{P_1}\cancel{\left(\frac{P_1}{P_2}\right)^{1/\gamma} \cdot V_1} \Big/ \cancel{\left(\frac{P_1}{P_2}\right)^{1/\gamma} \cdot V_1}\right]$$

$$\boxed{W = P_2 V_1 \left(\frac{P_1}{P_2}\right)^{1/\gamma} \ln\left(\frac{P_2}{P_1}\right)} \boxed{= Q}$$

$$\frac{P_3 V_3}{T_3} = \frac{P_1 V_1}{T_1}$$

$$V_3 = \frac{T_3}{T_1} V_1 = \frac{T_2}{T_1} V_1$$

$$= \frac{P_2}{P_1}\left(\frac{P_1}{P_2}\right)^{1/\gamma} \cdot V_1$$

(c) Isobaric process $3 \to 1$

$$W = \int_{V_3}^{V_1} p\,dV = p_1 \int_{V_3}^{V_1} dV = p_1 (V_1 - V_3) = p_1 \left(V_1 - \left(\frac{p_2}{p_1}\right)^{1-1/\gamma} V_1\right)$$

$$\boxed{W = p_1 V_1 \left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)}$$

$$\Delta E = \frac{3}{2} nR\,\Delta T = \frac{3}{2} nR (T_1 - T_3) = \frac{3}{2} nR \left(\frac{p_1 V_1}{nR} - \frac{p_2 V_1}{nR}\left(\frac{p_1}{p_2}\right)^{1/\gamma}\right)$$

$[T_2 = T_3]$

$$= \frac{3}{2} p_1 V_1 \left(1 - \frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{1/\gamma}\right)$$

by the first law $\Delta E = Q - W$

$Q = \Delta E + W$

$$\boxed{Q = \frac{3}{2} p_1 V_1 \left(1 - \frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{1/\gamma}\right) + p_1 V_1 \left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)}$$

(d) $\beta = \frac{1}{V}\frac{dV}{dT}$ for the isobar $p_1 = p_3$

$$\frac{p_1 V_1}{T_1} = \frac{p_3 V_3}{T_3} = \frac{p_1 V}{T}$$

$$\frac{V_1}{T_1} = \frac{V}{T}$$

$$V = \frac{V_1}{T_1} \cdot T = \frac{V_1}{\frac{p_1 V_1}{nR}} \cdot T = \frac{nRT}{p_1}$$

$$= \frac{p_1}{nRT} \cdot \frac{nR}{p_1}$$

$$\boxed{\beta = \frac{1}{T}}$$

4. a.

	1	2	3	4	5	6
1	1	1.5	2	2.5	3	3.5
2	1.5	2	2.5	3	3.5	4
3	2	2.5	3	3.5	4	4.5
4	2.5	3	3.5	4	4.5	5
5	3	3.5	4	4.5	5	5.5
6	3.5	4	4.5	5	5.5	6

If each die has equal chance of coming up each number, all microstates equally likely, thus

b.

macrostate	# microstates	entropy = $k_B \ln \Omega$
1	1	0
2.5	4	$k_B \ln 4 = 1.91 \times 10^{-23} \frac{J}{K}$
5	3	$k_B \ln 3 = 1.52 \times 10^{-23} \frac{J}{K}$

c. Largest number of microstates: 6,
macrostate 3.5

$$S = k_B \ln \Omega = 1.38 \times 10^{-23} \ln 6 = 2.47 \times 10^{-23} \frac{J}{K}$$

d. Since dice are rerolled, initial state won't matter. The most likely macrostate is that with the most microstates, since all microstates equally likely. Therefore, the final macrostate will most likely be 3.5