NAME ____________________________________________________________

CHEMICAL ENGINEERING 150A
Mid-term Examination 2

NOTES:

You may indicate on the equation sheets what terms you are neglecting (that is, **you do not need to copy the complete equations and then cross out terms, but may indicate the terms you are neglecting on the attached tables**.)

Be sure to **write your name on the equations sheet and turn it in**.

Also, **indicate next to the appropriate table which problem you are solving.**

**Write down all assumptions made.**

Show your work.

| Problem | Score |
|---|---|
| 1 | /45 |
| 2 | /35 |
| 3 | /20 |
| Total | /100 |

Name_______________________________________________________

1. (45 pts total) Consider the axial annular flow of an incompressible power-law fluid as shown in the figure below. Fluid in the reservoir on the left is drawn through the annular region between the rod of radius bR and the cylinder of radius R by the motion of the rod, which moves with an axial velocity V and is concentric with the outer cylinder. The fluid in both the upstream and downstream reservoirs is at pressure $p_0$ and the length L is much greater than R or bR.

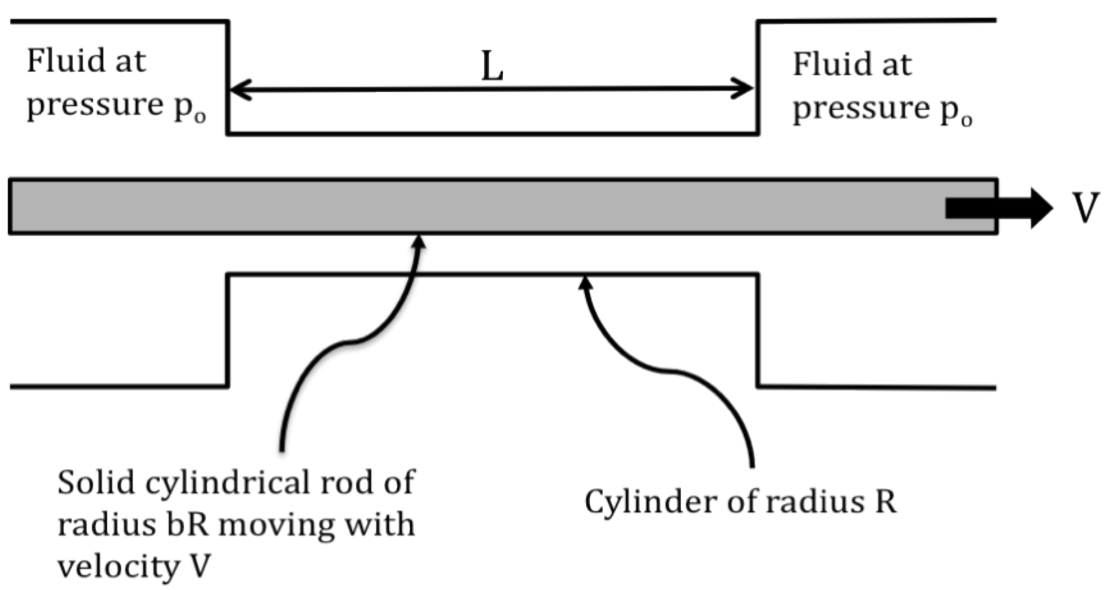
For a power-law fluid the viscosity is given by:

$$\eta = K\left|\frac{1}{2}II\right|^{(n-1)/2}$$

where K and n are constants.

a) (30 pts) Solve for the velocity field for the power-law fluid in the annular region, neglecting entrance and end effects.
b) (10 pts) Calculate the drag force acting on the rod over the region between the two reservoirs.
c) (5 pts) Set up the equation to calculate the volumetric flow rate in the central channel. You do not need to simplify or solve the equation, but be sure that everything is specified so that someone could solve it.

NOTE: If you cannot do the power-law fluid problem, the corresponding calculations for a Newtonian fluid will be worth a total of 30 points of the 45 possible.



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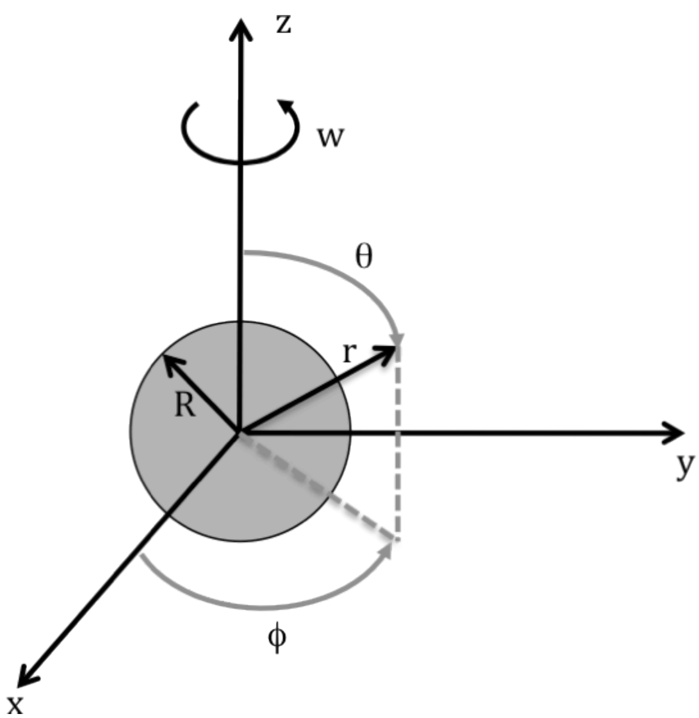
2. (35 pts) A solid sphere of radius R is slowly rotating at an angular velocity w about the z axis, as shown in the figure below. (The coordinates, indicated below, are the same as those used in other problems involving spheres). The sphere is immersed in a large volume of fluid, and the fluid is stagnant far from the sphere. Assume that the rotation is sufficiently slow that Re <<1.

a) Ultimately, we want the velocity field. Start by writing down your assumptions, including the assumed dependence of the velocity field on r, θ, and ϕ. Show that this form is consistent with the continuity equation.
b) Recalling how we solved creeping flow problems in class, write the boundary conditions, assume a form for the velocity field and derive a differential equation using your assumed form for the velocity field.
c) Solve the differential equation for the velocity field subject to the boundary conditions.
The mathematical hint below may be useful:

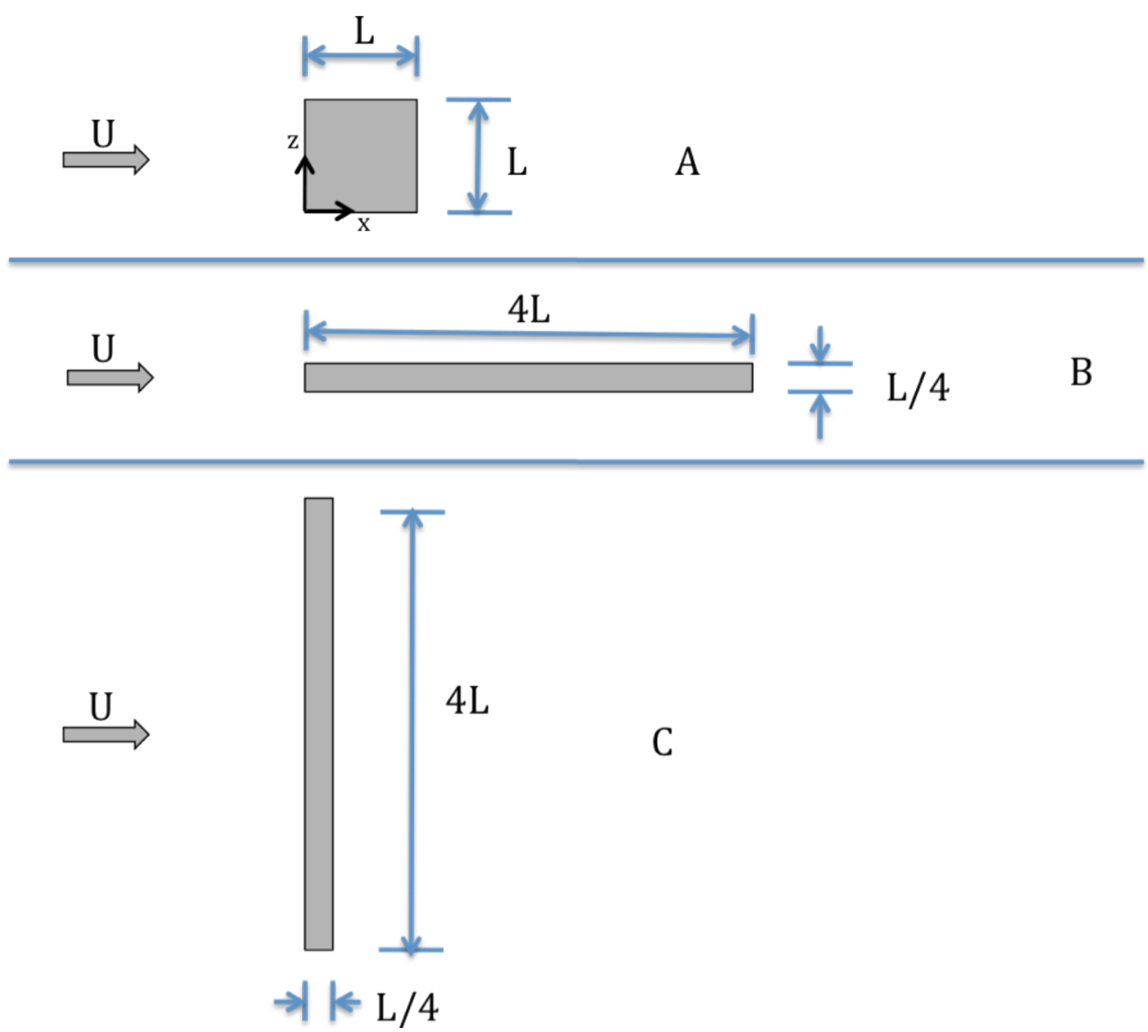
$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0$$

where a, b, and c are constants, has solutions of the form: $y = x^n$.



Name ___________________________________________

3. (20 pts) The three flat plates (below are **top views** of the plates) are oriented parallel to the free stream as indicated. All three have the same area.
a) Calculate the total drag force on the top of the plates A, B, and C assuming the boundary layers remain laminar in all cases and compare the values. **Use coordinates labeled as in A in your answer.**
b) Explain your answer (that is, the comparison) physically.



## TABLE 7-1
## CONTINUITY EQUATION IN RECTANGULAR, CYLINDRICAL, AND SPHERICAL COORDINATES

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

Rectangular $(x, y, z)$ coordinates:

$$\nabla \cdot (\rho \mathbf{v}) = \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)$$

Cylindrical $(r, \theta, z)$ coordinates:

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z)$$

Spherical $(r, \theta, \phi)$ coordinates:

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi)$$

## TABLE 7-5
## CAUCHY MOMENTUM EQUATION IN CYLINDRICAL $(r, \theta, z)$ COORDINATES

$r$ component:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} + \rho g_r$$

$\theta$ component:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \rho g_\theta$$

$z$ component:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z$$

## TABLE 7-6
## STRESS CONSTITUTIVE EQUATION FOR A NEWTONIAN FLUID IN CYLINDRICAL $(r, \theta, z)$ COORDINATES

$$\tau_{rr} = \eta\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{\theta\theta} = \eta\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{zz} = \eta\left[2\frac{\partial v_z}{\partial z} - \tfrac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \eta\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\tau_{zr} = \tau_{rz} = \eta\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

## TABLE 7-7
## NAVIER–STOKES EQUATIONS FOR A NEWTONIAN FLUID WITH A CONSTANT VISCOSITY IN CYLINDRICAL $(r, \theta, z)$ COORDINATES[a]

$r$ component:
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial \mathcal{P}}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$\theta$ component:
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial \mathcal{P}}{\partial \theta} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$z$ component:
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial \mathcal{P}}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

[a]The equations are written in terms of the equivalent pressure, $\mathcal{P}$.

TABLE 7-8
CAUCHY MOMENTUM EQUATION
IN SPHERICAL $(r, \theta, \phi)$ COORDINATES

$r$ component:
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{r\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial\phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} + \rho g_r$$

$\theta$ component:
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi} + \rho g_\theta$$

$\phi$ component:
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\phi}) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi} + \rho g_\phi$$

TABLE 7-9
STRESS CONSTITUTIVE EQUATION FOR A NEWTONIAN FLUID
IN SPHERICAL $(r, \theta, \phi)$ COORDINATES

$$\tau_{rr} = \eta\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{\theta\theta} = \eta\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{\phi\phi} = \eta\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right) - \tfrac{2}{3}(\nabla \cdot \mathbf{v})\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \eta\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \eta\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

## TABLE 7-10

## NAVIER–STOKES EQUATIONS FOR A NEWTONIAN FLUID WITH A CONSTANT VISCOSITY IN SPHERICAL $(r, \theta, \phi)$ COORDINATES[a]

$r$ component:
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial \mathcal{P}}{\partial r} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right.$$
$$\left. - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2}v_\theta\cot\theta - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right]$$

$\theta$ component:
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial \mathcal{P}}{\partial \theta} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}\right]$$

$\phi$ component:
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial \mathcal{P}}{\partial \phi} + \eta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}\right.$$
$$\left. - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\right]$$

[a]The equations are written in terms of the equivalent pressure, $\mathcal{P}$.

## TABLE 8-1

## THE FUNCTION $\frac{1}{2}\Pi$ IN RECTANGULAR, CYLINDRICAL, AND SPHERICAL COORDINATES

Rectangular:
$$\tfrac{1}{2}\Pi = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right]$$
$$+ \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]^2$$

Cylindrical:
$$\tfrac{1}{2}\Pi = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right]$$
$$+ \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial\theta} + \frac{\partial v_\theta}{\partial z}\right]^2$$
$$+ \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2$$

Spherical:
$$\tfrac{1}{2}\Pi = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2\right.$$
$$\left. + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2\right]$$
$$+ \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]^2$$
$$+ \left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]^2$$
$$+ \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2$$