Final Examination
Thursday August 9, 2012
10:00 am to 12:30 pm
170 Barrows Hall

Closed Books and Closed Notes
Each Question is Worth 25 Points

Useful Formulae

For all the corotational bases shown in the figures

$$\begin{aligned} \mathbf{e}_x &= \cos(\theta)\mathbf{E}_x + \sin(\theta)\mathbf{E}_y, \\ \mathbf{e}_y &= \cos(\theta)\mathbf{E}_y - \sin(\theta)\mathbf{E}_x. \end{aligned} \tag{1}$$

The following identity for the angular momentum of a rigid body relative to a point P will also be useful:

$$\mathbf{H}_P = \mathbf{H} + (\bar{\mathbf{x}} - \mathbf{x}_P) \times m\bar{\mathbf{v}}. \tag{2}$$

In computing components of moments, the following identity can be useful:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{E}_z = (\mathbf{E}_z \times \mathbf{a}) \cdot \mathbf{b}. \tag{3}$$

Finally, recall that the work-energy theorem of a rigid body which is subject to a system of K forces and a pure moment $\mathbf{M}_p$ is

$$\dot{T} = \sum_{i=1}^{K} \mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_p \cdot \boldsymbol{\omega}. \tag{4}$$

Here, $\mathbf{v}_i$ is the velocity vector of the point where the force $\mathbf{F}_i$ is applied.

Question 1

A Particle in a Slotted Disk

As shown in Figure 1, a particle of mass m_1 is attached by a pair of identical springs to a circular disk of mass m_2. The particle is free to move in a smooth grove milled on the surface of the circular disk. The disk is fixed at its center point O and is free to rotate about the vertical $\mathbf{E}_z$ axis.

The position vector of the particle relative to the center O of the disk is

$$\mathbf{x}_1 = h\mathbf{e}_y + x\mathbf{e}_x \tag{5}$$

where h is constant. The moment of inertia of the disk is I_{Ozz}, and the stiffnesses of the springs are $\frac{K}{2}$.

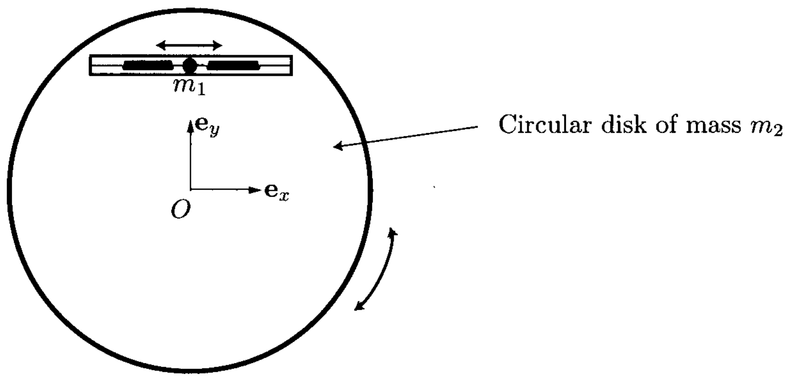


Figure 1: *A particle of mass m_1 is free to move in a smooth slot on a circular disk of mass m_2.*

(a) *(10 Points)* Show that the kinetic energy and angular momentum relative to O of the particle-disk system is

$$\begin{aligned} \mathbf{H}_O &= H_O\mathbf{E}_z = \left(I_{Ozz} + m_1\left(h^2 + x^2\right)\right)\dot{\theta}\mathbf{E}_z - m_1 h\dot{x}\mathbf{E}_z, \\ T &= \frac{m_1}{2}\dot{x}^2 + \frac{1}{2}\left(I_{Ozz} + m_1\left(h^2 + x^2\right)\right)\dot{\theta}^2 - m_1 h\dot{x}\dot{\theta}. \end{aligned} \tag{6}$$

(b) *(5 Points)* Draw a free-body diagram of the particle and a free-body diagram of the circular disk.

(c) *(5 Points)* Show that the equations governing the motion of the disk-particle system are

$$\frac{dH_O}{dt} = 0, \qquad m_1\left(\ddot{x} - h\ddot{\theta} - x\dot{\theta}^2\right) = -Kx. \tag{7}$$

(d) *(5 Points)* Give an expression for the total energy E of the system and verify with the help of (7), that $\dot{E} = 0$.

Question 2

A Sliding Rigid Body

As shown in Figure 2, a circular cylinder of radius R, mass m, and moment of inertia (relative to its center of mass) I_{zz} slides on a rough inclined plane. The position vector of the center of mass C has the representation

$$\bar{\mathbf{x}} = x\mathbf{E}_x + y_0\mathbf{E}_y, \tag{8}$$

where y_0 is a constant. The point P is the instantaneous point of contact.

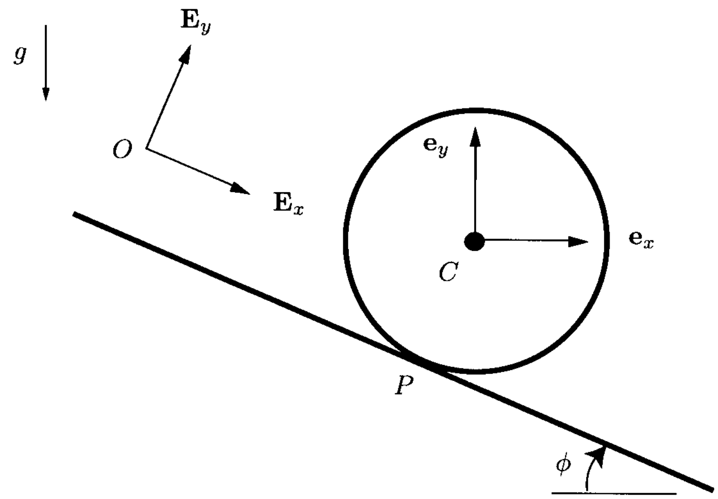


Figure 2: *A rigid cylinder of mass m and radius R sliding on an inclined plane.*

(a) *(5 Points)* Using the identity $\mathbf{v}_2 = \mathbf{v}_1 + \boldsymbol{\omega} \times (\mathbf{x}_2 - \mathbf{x}_1)$ applied to two points on the rigid body, show the velocity vector of the instantaneous point of contact has the representation

$$\mathbf{v}_P = v_P\mathbf{E}_x, \qquad v_P = \dot{x} + R\dot{\theta}, \tag{9}$$

where $\boldsymbol{\omega} = \dot{\theta}\mathbf{E}_z$ is the angular velocity of the rigid body.

(b) *(5 Points)* Draw a free-body diagram of the rigid body. In your free-body diagram give a clear expression for the friction force.

(c) *(6 Points)* Using balances of linear and angular momentum, show that the differential equation governing the slip velocity v_P is

$$\dot{v}_P = g\sin(\phi) - \mu_d g\cos(\phi)\left(\frac{I_{zz} + mR^2}{I_{zz}}\right)\frac{v_P}{|v_P|}, \tag{10}$$

where μ_d is the coefficient of dynamic friction.

(d) *(4 Points)* Starting from the work-energy theorem (4), prove that the total energy E of the sliding rigid body decreases with time.

(e) *(5 Points)* Suppose $\phi = 0$ and that the body is given an initial velocities $\dot{x}_0$ and $\dot{\theta}_0$ where $v_{P_0} = \dot{x}_0 + R\dot{\theta}_0 < 0$. Determine the time τ that it will take for the slip velocity to reach zero.

Question 3

A Tipping Point

As shown in Figure 3, a slender rod of mass m, length L, and moment of inertia $I_{zz} = \frac{mL^2}{12}$ is at rest in a vertical position with the material point labelled A in contact with a rough horizontal surface. The rod is given a slight inclination and released. As the rod falls, the friction force at A eventually become insufficient to prevent the point A from sliding.

During its motion, a vertical gravitational force $-mg\mathbf{E}_y$ acts on the rod.

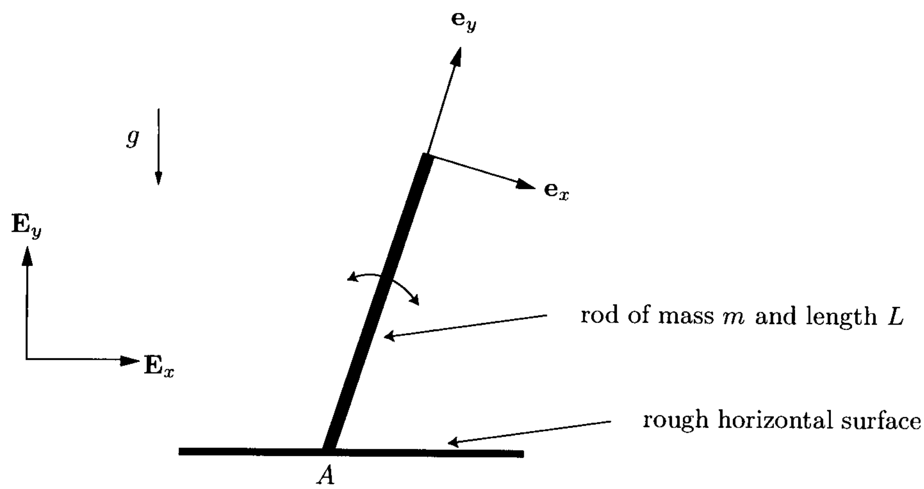


Figure 3: *A slender rod of mass m is free to rotate about A and the point A is free to move on a rough horizontal surface.*

The position vector of the center of mass C has the representation

$$\bar{\mathbf{x}} = \frac{L}{2}\mathbf{e}_y + \mathbf{x}_A, \tag{11}$$

where $\mathbf{x}_A$ is the position vector of A.

(a) *(5 Points)* Show that the angular momentum $\mathbf{H}_A$ of the rod and the kinetic energy of the rod have the representations

$$\mathbf{H}_A = I_{Azz}\dot{\theta}\mathbf{E}_z - \frac{mLv_A}{2}\cos(\theta)\,\mathbf{E}_z, \qquad T = \frac{1}{2}I_{Azz}\dot{\theta}^2 + \frac{m}{2}v_A^2 - \frac{mv_AL}{2}\dot{\theta}\cos(\theta), \tag{12}$$

where $I_{Azz} = \frac{mL^2}{3}$ and $\dot{\mathbf{x}}_A = v_A\mathbf{E}_x$.

(b) *(5 Points)* Assuming that A is in contact with the horizontal surface, draw a free-body diagram of the rod. Distinguish the cases where $\mathbf{v}_A = \mathbf{0}$ and $\mathbf{v}_A \neq \mathbf{0}$.

(c) *(10 Points)* Assuming that A is stationary, show that

$$I_{Azz}\ddot{\theta} = \frac{mgL}{2}\sin(\theta), \quad \mathbf{R}_A = mg\mathbf{E}_y - \frac{mL}{2}\left(\ddot{\theta}\mathbf{e}_x + \dot{\theta}^2\mathbf{e}_y\right) \tag{13}$$

where $\mathbf{R}_A$ is the reaction force at A. In addition, using the work-energy theorem (4), prove that the total energy of the rod is conserved. In your solution give a clear expression for the total energy E.

(d) *(5 Points)* Suppose A is sliding on the horizontal surface. Outline how you would determine the equations governing the motion of the rod.

Question 4

A Particle Colliding with a Rigid Body

As shown in Figure 4, a particle of mass m_1 is traveling with a velocity $-v_0\mathbf{E}_y$ when it collides with a stationary rod. The uniform rod, which has a mass m_2, length L, and a moment of inertia relative to its center of mass of I_{zz}, is free to move on a horizontal plane. Following the collision, the particle adheres to the rod and the particle-rod move as a single rigid body on the smooth horizontal plane.

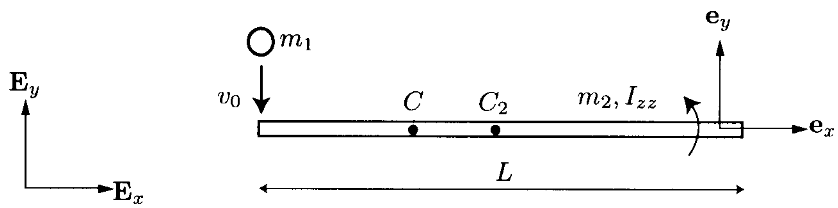


Figure 4: *A particle of mass m_1 collides with a rod of mass m_2. The basis vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{E}_z\}$ corotate with the rod*

(a) *(4 Points)* Assuming that the particle adheres to the end of the rod following the collision, show that the position vector $\bar{\mathbf{x}}$ of the center of mass C of the particle-rod following the collision has the representation

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}_2 - \left(\frac{m_1}{m_1 + m_2}\right)\frac{L}{2}\mathbf{e}_x \tag{14}$$

where $\bar{\mathbf{x}}_2$ is the position vector of the center of mass C_2 of the rod.

(b) *(5 Points)* Assuming that the particle adheres to the end of the rod following the collision, determine the velocity vector of the center of mass of the particle-rod immediately following the collision.

(c) *(8 Points)* Show that the angular velocity vector $\boldsymbol{\omega}$ of the particle-rod immediately following the collision can be expressed in the form

$$\omega\mathbf{E}_z = \left(\frac{m_1 m_2}{m_1 + m_2}\right)\left(\frac{Lv_0}{2I}\right)\mathbf{E}_z. \tag{15}$$

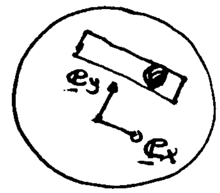
For full credit, provide an expression for I.

(d) *(8 Points)* Show that the kinetic energy lost during the collision can be expressed in the form

$$\Delta T = \frac{1}{2}m_1 v_0^2 \left(1 - H\right). \tag{16}$$

For full credit, provide an expression for H.

QUESTION 1



(a) $\quad \underline{x}_1 = h\underline{e}_y + x\underline{e}_x$

$\quad \dot{\underline{x}}_1 = -h\dot{\theta}\underline{e}_x + \dot{x}\underline{e}_x + x\dot{\theta}\underline{e}_y$

$$\underline{H}_o = I_{Ozz}\dot{\theta}\underline{E}_z + \underline{x}_1 \times m_1 \dot{\underline{x}}_1$$

$$= (I_{Ozz}\dot{\theta})\underline{E}_z + m_1(h\underline{e}_y + x\underline{e}_x) \times ((\dot{x} - h\dot{\theta})\underline{e}_x + x\dot{\theta}\underline{e}_y)$$

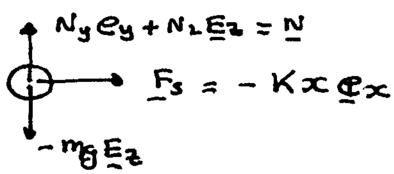
$$= (I_{Ozz} + m_1(h^2 + x^2))\dot{\theta}\underline{E}_z - m_1 h\dot{x}\underline{E}_z$$

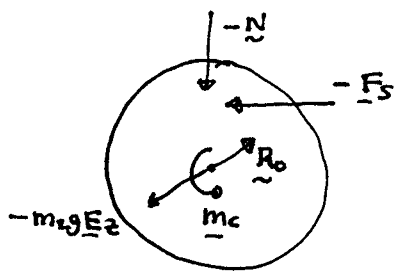
$$T = \frac{1}{2} I_{Ozz}\dot{\theta}^2 + \frac{1}{2} m_1 \dot{\underline{x}}_1 \cdot \dot{\underline{x}}_1$$

$$= \frac{1}{2} I_{Ozz}\dot{\theta}^2 + \frac{1}{2} m_1 ((\dot{x}^2 + h^2\dot{\theta}^2 - 2h\dot{\theta}\dot{x}) + x^2\dot{\theta}^2)$$

$$= \frac{1}{2}(I_{Ozz} + m_1(x^2 + h^2))\dot{\theta}^2 - m_1 h\dot{\theta}\dot{x} + \frac{1}{2} m_1 \dot{x}^2$$

(b)





$\underline{m}_c = m_{cx}\underline{e}_x + m_{cy}\underline{e}_y$

$\underline{R}_o = R_{ox}\underline{E}_x + R_{oy}\underline{E}_y + R_{oz}\underline{E}_z$

(c) Balance of Angular momentum of the Disk-Particle system about O:

$$\frac{d\underline{H}_o}{dt} = \underline{m}_o$$

Taking the $\underline{E}_z$ component of this equation $\quad \underline{m}_o \cdot \underline{E}_z = 0$

$$\Rightarrow \quad \frac{d}{dt}H_o = 0 \qquad \text{where} \quad H_o = \underline{H}_o \cdot \underline{E}_z$$

For the particle

$$\underline{F}_1 = m_1 \ddot{\underline{x}}_1 \qquad \ddot{\underline{x}}_1 = -h\ddot{\theta}\,\underline{e}_x - h\dot{\theta}^2\,\underline{e}_y + \ddot{x}\,\underline{e}_x + 2\dot{x}\dot{\theta}\,\underline{e}_y - x\dot{\theta}^2\,\underline{e}_x + x\ddot{\theta}\,\underline{e}_y.$$

Taking the $\underline{e}_x$ component of this equation

$$-Kx = m_1 \ddot{\underline{x}}_1 \cdot \underline{e}_x = m_1 (\ddot{x} - x\dot{\theta}^2 - h\ddot{\theta})$$

d) $$E = T + \tfrac{1}{2}Kx^2$$

$$\dot{E} = m_1\dot{x}\ddot{x} + (I_{Ozz} + m_1(h^2+x^2))\dot{\theta}\ddot{\theta} + m_1 x\dot{x}\dot{\theta}^2 - m_1 h\ddot{x}\dot{\theta} - m_1 h\dot{x}\ddot{\theta} + Kx\dot{x} \quad (*)$$

Substituting $Kx = -m_1(\ddot{x} - x\dot{\theta}^2 - h\ddot{\theta})$ and $\dot{H}_O = 0$ into $(*)$ yields the result $\dot{E} = 0$

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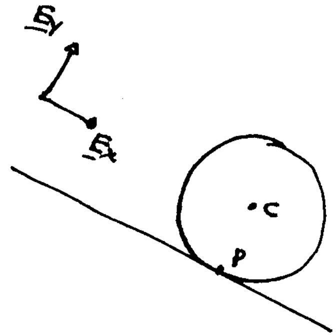
QUESTION 2



(a)

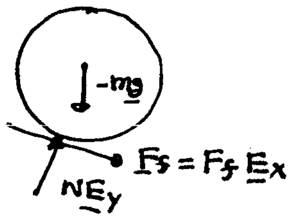
$$\underline{v}_P = \bar{\underline{v}} + \underline{\omega} \times (\underline{x}_P - \bar{\underline{x}})$$

$$= \dot{x}\underline{E}_x + \dot{\theta}\underline{E}_z \times (-R\underline{E}_y)$$

$$= (\dot{x} + R\dot{\theta})\underline{E}_x$$

$$= v_P \underline{E}_x$$

(b)



$$-m\underline{g} = mg\sin\phi\,\underline{E}_x - mg\cos\phi\,\underline{E}_y$$

$$\underline{F}_f = -\mu_d \|\underline{N}\| \frac{v_P}{|v_P|}\underline{E}_x$$

(c) $\underline{F} = m\ddot{\bar{\underline{x}}}$: $\cdot\underline{E}_x$ $\quad F_f + mg\sin\phi = m\ddot{x}$

$\cdot\underline{E}_y$ $\quad N = mg\cos\phi$

$\underline{M} = \dot{\underline{H}}$: $\cdot\underline{E}_z$ $\quad RF_f = I_{zz}\ddot{\theta}$

Hence

$$m\dot{v}_P = m\ddot{x} + m\ddot{\theta}R$$

$$= mg\sin\phi + F_f + \frac{mR^2}{I_{zz}}F_f$$

$$= mg\sin\phi + \left(1 + \frac{mR^2}{I_{zz}}\right)F_f$$

$$= mg\sin\phi + \left(1 + \frac{mR^2}{I_{zz}}\right) - \mu_d\, mg\cos\phi\, \frac{v_P}{|v_P|}$$

Rearranging

$$\dot{v}_P = g\sin\phi - \mu_d\left(1 + \frac{mR^2}{I_{zz}}\right) g\cos\phi\, \frac{v_P}{|v_P|}$$

(d)

$$\dot{T} = \underline{F}_f \cdot \underline{v}_P + \underline{N} \cdot \underline{v}_P - m\underline{g} \cdot \dot{\bar{\underline{x}}}$$

$$\Rightarrow \frac{d}{dt}\left(E = T + m\underline{g} \cdot \bar{\underline{x}}\right) = \underline{F}_f \cdot \underline{v}_P + \underline{N} \cdot \underline{v}_P$$

$$= -\mu_d \|\underline{N}\| \frac{\underline{v}_P \cdot \underline{v}_P}{\|\underline{v}_P\|} + 0 \quad (\infty\ \underline{N} \perp \underline{v}_P)$$

$$= -\mu_d \|\underline{N}\| \|\underline{v}_P\|$$

$$< 0$$

Hence E decreases with time.

(e) If $\phi = 0$ and $v_P(0) < 0$ then

$$\dot{v}_P = -\mu_d g \left(\frac{I_{22} + mR^2}{I_{22}}\right)(-1) \qquad \text{when} \quad v_P(t) < 0$$

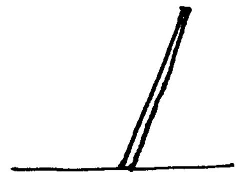
Hence integrating this equation and setting $v_P(\tau) = 0$

$$0 - (+v_P(0)) = +\mu_d g \left(\frac{I_{22} + mR^2}{I_{22}}\right)\tau$$

$$\Rightarrow \qquad \tau = \frac{-I_{22}\, v_P(0)}{I_{22} + mR^2}\, \frac{1}{\mu_d g.}$$

Note that $-v_P(0) > 0$
so $T > 0$.

QUESTION 3



(a)

$$\underline{H}_A = \underline{H} + (\bar{\underline{x}} - \underline{x}_A) \times m\bar{\underline{v}}$$

$$= I_{22}\dot{\theta}\underline{E}_z + \frac{L}{2}\underline{e}_y \times (m v_A \underline{E}_x - m\frac{L}{2}\dot{\theta}\underline{e}_x)$$

$$\bar{\underline{x}} = \underline{x}_A + \frac{L}{2}\underline{e}_y$$

$$\bar{\underline{v}} = \underline{v}_A + \frac{L}{2}\dot{\theta}\underline{e}_x$$

$$= I_{22}\dot{\theta}\underline{E}_z - \frac{mL}{2} v_A \cos\theta \underline{E}_z + \frac{mL^2}{4}\dot{\theta}\underline{E}_z$$

$$= \left(I_{22} + \frac{mL^2}{4}\right)\dot{\theta}\underline{E}_z - \frac{mL}{2} v_A \cos\theta \underline{E}_z$$

$$= I_{A22}\dot{\theta}\underline{E}_z - \frac{mL}{2} v_A \cos\theta \underline{E}_z$$

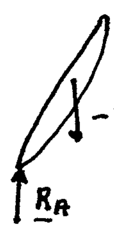
$$T = \frac{1}{2} m \bar{\underline{v}}\cdot\bar{\underline{v}} + \frac{1}{2}\underline{H}\cdot\underline{\omega}$$

$$= \frac{1}{2} m \left(v_A \underline{E}_x - \frac{L}{2}\dot{\theta}(\cos\theta \underline{E}_x + \sin\theta \underline{E}_y)\right)\cdot\bar{\underline{v}} + \frac{1}{2} I_{22}\dot{\theta}^2$$

$$= \frac{1}{2} m v_A^2 + \frac{1}{2} m\frac{L^2}{4}\dot{\theta}^2 - \frac{mL}{2}\dot{\theta}\cos\theta v_A + \frac{1}{2} I_{22}\dot{\theta}^2$$

$$= \frac{1}{2} m v_A^2 + \frac{1}{2} I_{A22}\dot{\theta}^2 - \frac{mL}{2}\dot{\theta} v_A \cos\theta$$

(b)



$$\underline{R}_A = N\underline{E}_y + F_f\underline{E}_x$$

$F_f\underline{E}_x$ is unknown when $\underline{v}_A = 0$

$$F_f\underline{E}_x = -\mu_d \|N\underline{E}_y\| \frac{v_A}{|v_A|}\underline{E}_x \quad \text{when } v_A \neq 0.$$

(c) When $\underline{V}_A = \underline{0}$ we can take moments about A

$$\dot{\underline{H}}_A = \underline{M}_A = +\frac{L}{2}\underline{e}_y \times -mg\underline{E}_y$$

$$= \frac{L}{2}(-\sin\theta\, \underline{E}_x + \cos\theta\, \underline{E}_y) \times -mg\underline{E}_y$$

$$= +\frac{mgL}{2}\sin\theta\, \underline{E}_z$$

Hence

$$I_{Azz}\ddot{\theta} = \frac{mgL}{2}\sin\theta$$

From $\underline{F} = m\dot{\underline{v}}$ where $\dot{\underline{v}} = -\frac{L}{2}\ddot{\theta}\,\underline{e}_x - \frac{L}{2}\dot{\theta}^2\underline{e}_y$

we find that

$$\underline{R}_A = +mg\underline{E}_y + m\dot{\underline{v}} = mg\underline{E}_y - \frac{mL}{2}(\ddot{\theta}\,\underline{e}_x + \dot{\theta}^2\underline{e}_y)$$

From Work-Energy theorem

$$\frac{dT}{dt} = \underbrace{\underline{R}_A \cdot \underline{V}_A}_{=0} - mg\underline{E}_y \cdot \dot{\bar{\underline{x}}} = -\frac{d}{dt}\left(mg\underline{E}_y \cdot \bar{\underline{x}}\right)$$

$$\Rightarrow \quad \frac{d}{dt}\left(E = T + mg\underline{E}_y \cdot \bar{\underline{x}}\right) = 0.$$

(d) If A is sliding then the equations governing the motion of the rod are found from $\underline{F} = m\dot{\underline{v}}$ and $\underline{M} = \dot{\underline{H}}$:

$$\underline{R}_A - mg\underline{E}_y = m(\dot{v}_A\underline{E}_x - \frac{L}{2}\ddot{\theta}\,\underline{e}_x - \frac{L}{2}\dot{\theta}^2\underline{e}_y)$$

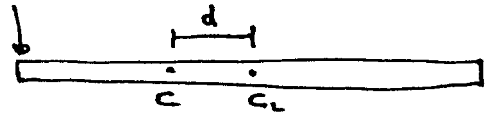
$$I_{zz}\ddot{\theta}\,\underline{E}_z = -\frac{L}{2}\underline{e}_x \times \underline{R}_A$$

These equations provide 3 equations for the unknowns N, θ and v_A

In these equations

$$\underline{R}_A = N\underline{E}_y - \mu_d N \frac{v_A}{|v_A|}\underline{E}_x$$

QUESTION 4



(a)

By definition $$(m_1+m_2)\underline{\bar{x}} = m_1\underline{x}_1 + m_2\underline{\bar{x}}_2$$

~~Taking the ori~~

However $$\underline{\bar{x}}_2 = \underline{x}_1 + \frac{L}{2}\underline{E}_x$$

so $$(m_1+m_2)\underline{\bar{x}} = m_1\underline{x}_2 + m_2\underline{x}_2 - \frac{m_1 L}{2}\underline{E}_x$$

and so
$$\underline{\bar{x}} = \underline{\bar{x}}_2 - \left(\frac{m_1}{m_1+m_2}\right)\frac{L}{2}\underline{E}_x$$

Consequently
$$\underline{\bar{x}} = \underline{x}_1 + \left(\frac{m_2}{m_1+m_2}\right)\frac{L}{2}\underline{E}_x$$

(b) Linear momentum of the system is conserved so

$$(m_1+m_2)\underline{\bar{v}} = -m_1 v_0 \underline{E}_y$$

Hence $$\underline{\bar{v}} = -\frac{m_1}{m_1+m_2} v_0 \underline{E}_y$$

(c) During collision $\underline{H}$ is conserved

Before collision $$\underline{H} = -(\underline{x}_1 - \underline{\bar{x}}) \times m_1 v_0 \underline{E}_y$$

$$= \left(\frac{m_2}{m_1+m_2}\right)\frac{L}{2}\underline{E}_x \times m_1 v_0 \underline{E}_y$$

$$= \frac{m_1 m_2 L v_0}{2(m_1+m_2)}\underline{E}_z$$

After collision $$\underline{H} = I_{zz}^{\text{system}}\dot{\theta}\underline{E}_z$$

$$I = I_{zz}^{\text{system}} = I_{zz} + \frac{m_2 L^2}{4}\frac{m_1^2}{(m_1+m_2)^2} + \frac{m_1 L^2}{4}\left(\frac{m_2^2}{(m_1+m_2)^2}\right)$$

$$= I_{zz} + \frac{m_1 m_2}{m_1+m_2}\frac{L^2}{4}$$

Using conservation of $\underline{H}\cdot\underline{E}_z$:

$$\omega = \frac{m_1 m_2 L v_0}{2(m_1+m_2)}\frac{1}{I}$$

(d) $$T_{before} = \frac{1}{2} m_1 V_0^2$$

$$T_{after} = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2}(m_1+m_2)\, \underline{V} \cdot \underline{V}$$

$$= \frac{1}{2} I \left(\frac{m_1 m_2 L V_0}{2(m_1+m_2)} \, \frac{1}{I} \right)^2$$

$$+ \frac{1}{2} (m_1+m_2) \left(\frac{m_1^2}{(m_1+m_2)^2} \right) V_0^2$$

$$= \frac{1}{2} V_0^2 m_1 \left(\frac{m_1 m_2^2 L^2}{2(m_1+m_2) I} + \frac{m_1}{m_1+m_2} \right) = \frac{1}{2} m_1 V_0^2 H$$

$$T_{before} - T_{after} = \frac{1}{2} m_1 V_0^2 \left(\frac{+m_2}{m_1+m_2} - \frac{m_1 m_2^2 L^2}{2(m_1+m_2) I} \right)$$

$$= \frac{1}{2} m_1 V_0^2 \left(\frac{m_2}{m_1+m_2} \right) \left(1 - \frac{m_1 m_2 L^2}{2 I} \right)$$ (one expression for ΔT)

$$= \frac{1}{2} m_1 V_0^2 \left(1 - H \right)$$ (alternative expression)