Physics 137A — Quantum Mechanics — Fall 2012
Midterm II - Solutions

These are the solutions to the exam given to Lecture 1.

Problem 1 [15 points]

Consider a particle with mass m and charge q in a simple harmonic oscillator potential.

(a) [4 points] If an externally applied electric field of magnitude E_{ext} is switched on in the region, sketch the new potential, $V(x) = V_{SHO}(x) + V_{elec}(x)$, where $V_{elec}(x) = qE_{ext}x$. Be sure to indicate important features of the potential on your sketch.

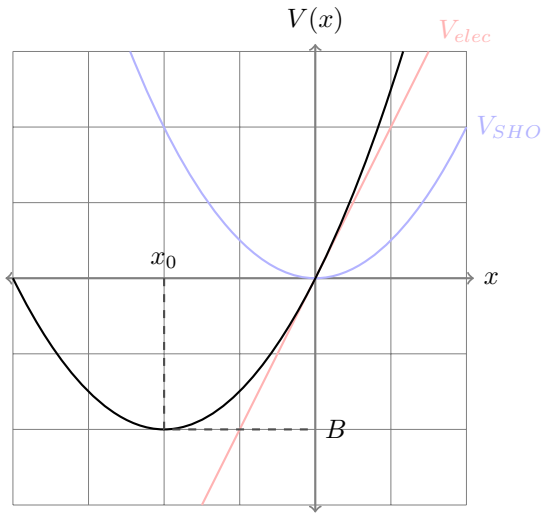
(b) [4 points] The potential can be written $V(x) = A(x - x_0)^2 + B$. What are A, x_0 and B? Feel free to use A, x_0 and B in the rest of this problem.

(c) [4 points] What are the allowed energies for this particle?

(d) [3 points] Write down the ground state wavefunction in this potential.

Solution

(a) The sketch is given below. There are three curves on the sketch: in black is the full potential of the system, in light blue is the V_{SHO} term, and in light red is the V_{elec} term.



(b) To find the coefficients A, B and x_0, just set the potential equal to the proposed form

$$\begin{aligned} \frac{1}{2}m\omega^2 x^2 + qE_{ext}x &= A(x - x_0)^2 + B \\ \frac{1}{2}m\omega^2 x^2 + qE_{ext}x &= Ax^2 + Ax_0^2 - 2Ax_0 x + B. \end{aligned}$$

Combining powers of x we find:

$$\left(\frac{1}{2}m\omega^2 - A\right)x^2 + (qE_{ext} + 2Ax_0)\,x - \left(B + Ax_0^2\right) = 0.$$

In order for this previous equation to be true, we need each term in parenthesis to vanish. So from the x^2 term we find

$$A = \frac{1}{2}m\omega^2.$$

From the x term we have

$$x_0 = -\frac{qE_{ext}}{2A} = -\frac{qE_{ext}}{m\omega^2}.$$

From the constant term we have

$$B = -Ax_0^2 = -\frac{q^2E_{ext}^2}{2m\omega^2}.$$

(c) As we saw in part (b), the potential for this system is still quadratic in x and still has the SHO form. The only change is that the minimum of the potential is no longer at $x = 0$, but at $x = x_0$, and the value of the potential at the minimum is not 0 but B.

The unmodified SHO problem gives energy levels $E_n = \hbar\omega(n + 1/2)$. Really, these are the energies above the minimum of the potential. So really it is $E_n - V_{min} = \hbar\omega(n + 1/2)$. In the modified potential the minimum of the potential is B. So, for the modified problem, the energies are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{q^2E_{ext}^2}{2m\omega^2}.$$

(d) The ground state wavefunction in this potential will match that of the unmodified SHO, but shifted by x_0. So the ground state will be

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x-x_0)^2}.$$

As we found in part (b), $x_0 = -qE_{ext}/(m\omega^2)$ so

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x+\frac{qE_{ext}}{m\omega^2})^2}.$$

Problem 2 [15 points]

A certain particle in a simple harmonic oscillator potential has the initial wave function

$$\Psi(x,0) = A\frac{1}{1+\frac{m\omega}{\hbar}x^2},$$

where A is a normalization constant (which you *do not* have to compute).

(a) [6 points] If you were to measure the energy of this particle, what values could you get? Briefly explain your answer.

(b) [6 points] Let the n-th energy eigenfunction of the simple harmonic oscillator be $\psi_n(x)$. The wavefunction, $\Psi(x,0)$, can be expanded in terms of these eigenfunctions. Write down this expansion and explain how you would compute the expansion coefficients, setting up any integrals you might need. You do not actually have to compute these coefficients.

(c) [3 points] Which of the energy eigenvalues is the most likely result of measuring the energy on this state? Briefly explain.

Solution

(a) The given wavefunction is even in x. The probability of getting a certain energy result is given by the inner product

$$P(E_n) = |\langle n|\Psi(x,0)\rangle|^2.$$

This inner product is equivalent to an integral over the energy eigenfunction times the initial conditions. The initial conditions are even in x, so it is only when the energy eigenfunction is also even

in x that their product is an even function. So the only possible cases are when n is even (including 0).

If instead n was odd, then the energy eigenfunction would be odd, and the product with the even initial conditions would be an odd function. This gives zero when integrated over the whole real line.

The possible results of measuring the energy of these initial conditions are all the energies $E_n = \hbar\omega(n + 1/2)$ where n is even.

(b) The expansion in terms of the energy eigenfunctions is given by

$$\Psi(x, 0) = \sum_{n=0}^{\infty} c_n \psi_n(x),$$

where c_n are complex expansion coefficients. To compute the numbers c_n, one performs the following inner product

$$c_n = \langle \psi_n | \Psi(0) \rangle = \int_{-\infty}^{\infty} \psi_n^*(x) \Psi(x, 0) dx.$$

Plugging in the initial conditions and the energy eigenfunctions we find that

$$c_n = \left(\frac{m\omega}{\pi\hbar}\right) \frac{A}{\sqrt{2^n n!}} \int_{-\infty}^{\infty} \frac{H_n(\sqrt{\frac{m\omega}{\hbar}}x) e^{-\frac{m\omega}{2\hbar}x^2}}{1 + \frac{m\omega}{\hbar}x^2}$$

where $\xi = \sqrt{m\omega/\hbar}x$, and $H_n(\xi)$ is the n-th Hermite polynomial.

(c) The most likely result is the ground state energy, $E_0 = \hbar\omega/2$. The reason for this is that each higher level has some regions where the product of $\Psi(x, 0)$ and $\psi_n(x)$ is positive and some regions where it is negative. The result is that unlike the ground state, which is an integral over an all positive product, the higher energy levels are integrals over both positive and negative values. This will have to give smaller integrals, and thus smaller probabilities.

Problem 3 [30 points]

Consider a two level system with the following Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & E_1 \\ E_1 & 0 \end{pmatrix}$$

where E_1 is a positive real constant (with units of energy). The system also has two observable quantities A and B represented by the following matrices:

$$\mathbf{A} = \begin{pmatrix} \hbar & -\hbar \\ -\hbar & \hbar \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} \hbar & i\hbar \\ -i\hbar & \hbar \end{pmatrix}.$$

(a) [10 points] At time $t = 0$, the system begins in the state

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the state of the system at times $t > 0$.

(b) [5 points] Which of the two observables A or B commutes with the Hamiltonian?

(c) [10 points] If the observable you selected in part (b) is measured at time $t = T$, what are the possible results of the measurement, and with what probability would you get each result.

(d) [5 points] Suppose that the measurement of the operator you selected in part (b) at time T gave the larger of the two possible measurement results. The system is left to evolve until $t = 3T$. What are the possible results of measuring the *energy* at $t = 3T$, and with what probability will you get each?

Solution

(a) To find the evolved version of the initial conditions, we will need the eigenvalues and eigenvectors of the Hamiltonian. To find these we need to find those values of λ that make the following determinant zero

$$\begin{vmatrix} -\lambda & E_1 \\ E_1 & -\lambda \end{vmatrix} = 0.$$

This gives the characteristic equation

$$\lambda^2 - E_1^2 = 0.$$

The solutions are easy to find. We will call these E_+ and E_-, where

$$E_\pm = \pm E_1.$$

The eigenvectors are also straightforward. If we call $\mathbf{\Psi}_+$ the eigenvector for E_+ and $\mathbf{\Psi}_-$ the eigenvector for E_- we must have

$$\mathbf{H}\mathbf{\Psi}_+ = E_1\mathbf{\Psi}_+$$

and

$$\mathbf{H}\mathbf{\Psi}_- = -E_1\mathbf{\Psi}_-.$$

It is easy to check that

$$\mathbf{\Psi}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{\Psi}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are normalized and orthogonal eigenvectors with the appropriate eigenvalue.

The next step in solving this part of the problem is to expand our initial conditions in these eigenvectors. Once we have done so, then the time dependence amounts to tacking on an exponential phase factor on each term. We are after an expression of the form:

$$\mathbf{\Psi}(0) = c_+\mathbf{\Psi}_+ + c_-\mathbf{\Psi}_-$$

where $c_\pm$ are complex numbers (the expansion coefficients). To find these coefficients, we can take inner products of the eigenvectors with the initial conditions. Thus, for c_+ we have

$$c_+ = \mathbf{\Psi}_+^\dagger\mathbf{\Psi}(0) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

and for c_- we have

$$c_- = \mathbf{\Psi}_-^\dagger\mathbf{\Psi}(0) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}.$$

This lets us write that

$$\mathbf{\Psi}(0) = \frac{1}{\sqrt{2}}\mathbf{\Psi}_+ + \frac{1}{\sqrt{2}}\mathbf{\Psi}_-.$$

To add the time dependence, we need to add in the exponential phase factors

$$\mathbf{\Psi}(t) = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}\mathbf{\Psi}_+ + \frac{1}{\sqrt{2}}e^{iE_1t/\hbar}\mathbf{\Psi}_-.$$

Written out in matrix form we have

$$\mathbf{\Psi}(t) = \frac{1}{2}\begin{pmatrix} e^{-iE_1t/\hbar} + e^{iE_1t/\hbar} \\ e^{-iE_1t/\hbar} - e^{iE_1t/\hbar} \end{pmatrix}.$$

We can simplify this with Euler's formula to find:

$$\mathbf{\Psi}(t) = \begin{pmatrix} \cos\left(\frac{E_1t}{\hbar}\right) \\ -i\sin\left(\frac{E_1t}{\hbar}\right) \end{pmatrix}.$$

(b) The answer to this question can be worked out by direct computation. Starting with $\mathbf{A}$ first we have

$$\begin{aligned} [\mathbf{H},\mathbf{A}] &= \mathbf{HA} - \mathbf{AH} \\ &= \begin{pmatrix} 0 & E_1 \\ E_1 & 0 \end{pmatrix}\begin{pmatrix} \hbar & -\hbar \\ -\hbar & \hbar \end{pmatrix} - \begin{pmatrix} \hbar & -\hbar \\ -\hbar & \hbar \end{pmatrix}\begin{pmatrix} 0 & E_1 \\ E_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\hbar E_1 & \hbar E_1 \\ \hbar E_1 & -\hbar E_1 \end{pmatrix} - \begin{pmatrix} -\hbar E_1 & \hbar E_1 \\ \hbar E_1 & -\hbar E_1 \end{pmatrix} \\ &= 0. \end{aligned}$$

So $\mathbf{A}$ is the operator that commutes with the Hamiltonian.

If you try this with the other you find

$$\begin{aligned} [\mathbf{H},\mathbf{B}] &= \mathbf{HB} - \mathbf{BH} \\ &= \begin{pmatrix} 0 & E_1 \\ E_1 & 0 \end{pmatrix}\begin{pmatrix} \hbar & i\hbar \\ -i\hbar & \hbar \end{pmatrix} - \begin{pmatrix} \hbar & i\hbar \\ -i\hbar & \hbar \end{pmatrix}\begin{pmatrix} 0 & E_1 \\ E_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -i\hbar E_1 & \hbar E_1 \\ \hbar E_1 & i\hbar E_1 \end{pmatrix} - \begin{pmatrix} i\hbar E_1 & \hbar E_1 \\ \hbar E_1 & -i\hbar E_1 \end{pmatrix} \\ &= \begin{pmatrix} -2i\hbar E_1 & 0 \\ 0 & 2i\hbar E_1 \end{pmatrix}. \end{aligned}$$

which is not zero.

(c) Because the operator $\mathbf{A}$ commutes with the Hamiltonian, they have simultaneous eigenvectors. So the eigenvectors we got in part (a) can be reused here. Let's just act this operator on the $\mathbf{\Psi}_\pm$ states to see what the eigenvalues are:

$$\mathbf{A}\mathbf{\Psi}_+ = \begin{pmatrix} \hbar & -\hbar \\ -\hbar & \hbar \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \hbar - \hbar \\ -\hbar + \hbar \end{pmatrix} = 0 = 0\mathbf{\Psi}_+$$

and

$$\mathbf{A}\mathbf{\Psi}_- = \begin{pmatrix} \hbar & -\hbar \\ -\hbar & \hbar \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2\hbar \\ -2\hbar \end{pmatrix} = 2\hbar\mathbf{\Psi}_-.$$

So the eigenvalues of the operator $\mathbf{A}$ are 0 and $2\hbar$ with corresponding eigenvectors $\mathbf{\Psi}_+$ and $\mathbf{\Psi}_-$. Thus, the possible results of measuring $\mathbf{A}$ on the state are 0 and $2\hbar$.

The probability of getting each result is the inner product of the eigenvector for the result with the state of the system. So

$$P(0) = \left|\mathbf{\Psi}_+^\dagger \mathbf{\Psi}(t)\right|^2 = \left|\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \cos\left(\frac{E_1t}{\hbar}\right) \\ -i\sin\left(\frac{E_1t}{\hbar}\right) \end{pmatrix}\right|^2 = \frac{1}{2}\left|e^{-iE_1t/\hbar}\right|^2 = \frac{1}{2}$$

and

$$P(2\hbar) = \left|\mathbf{\Psi}_-^\dagger \mathbf{\Psi}(t)\right|^2 = \left|\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{E_1 t}{\hbar}\right) \\ -i\sin\left(\frac{E_1 t}{\hbar}\right) \end{pmatrix}\right|^2 = \frac{1}{2}\left|e^{iE_1 t/\hbar}\right|^2 = \frac{1}{2}.$$

There is a 50% chance that the measurement would yield 0 and a 50% chance that the measurement would yield $2\hbar$.

(d) If the measurement at time T gave the larger value ($2\hbar$), then the state of the system collapses to being equal to the eigenvector corresponding to this eigenvalue. So at time $t = T$, the state will be

$$\mathbf{\Psi}(T) = \mathbf{\Psi}_-.$$

If this state is left to evolve forward in time starting at time $t = T$, we just have to expand this in energy eigenfunctions, and then add the exponential time dependence. This is, however, already an eigenvector of the Hamiltonian, so we are basically done. The way to add the time dependence so that we match the condition at time $t = T$ is the following

$$\mathbf{\Psi}(t > T) = e^{iE_1(t-T)/\hbar}\mathbf{\Psi}_-.$$

Now, if we measure the energy at time $t = 3T$, we just need to find the expansion of the state at that time in energy eigenfunctions, and square the expansion coefficients to find the probability of a given result. But once again, we are already written as an expansion in energy eigenfunctions. The expansion actually includes only the eigenvector for the eigenvalue of energy equal to $-E_1$. So $P(E_1) = 0$ and $P(-E_1) = 1$. So the only possible result at this time is to find $-E_1$.