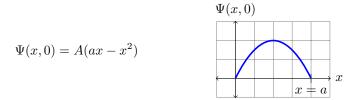# Physics 137A — Quantum Mechanics — Fall 2012
Midterm I - Solutions

1   Infinite Square Well Wavefunction

Consider the infinite square well of width a (extending from $x = 0$ to $x = a$). Take the initial spatial wave function to be a parabola centered around the middle of the well:

$$\Psi(x, 0)$$

$$\Psi(x, 0) = A(ax - x^2)$$

where A is some constant, a is the width of the well, and where this function applies only inside the well (outside the well, $\Psi(x, 0) = 0$).

(a) Normalize this wavefunction.

(b) Calculate the expectation values $\langle p \rangle$ and $\langle p^2 \rangle$ in this state.

(c) Given that $\langle x \rangle = a/2$ and $\langle x^2 \rangle = 2a^2/7$, what are the uncertainties σ_x and σ_p? Is the uncertainty principle satisfied?

(d) We can write this wavefunction as a sum over the energy eigenfunctions:

$$\Psi(x, 0) = \sum_{i=1}^{\infty} c_n \psi_n(x)$$

where c_n are some constants and $\psi_n(x)$ are the energy eigenfunctions. Set up a calculation of the coefficients, c_n, and write it as an integral with everything spelled out explicitly; you do not have to evaluate the integral. Are all of the c_n non-zero? Write a short argument justifying your answer.

Solution

(a) The normalization condition for this wavefunction is the following

$$1 = \int_0^a \Psi^*(x, 0)\Psi(x, 0)dx$$

Plugging in our wavefunction gives us

$$1 = |A|^2 \int_0^a (ax - x^2)^2 dx = |A|^2 \int_0^a (a^2x^2 - 2ax^3 + x^4)dx.$$

These are straightforward integrations to perform so we find

$$
\begin{aligned}
1 &= |A|^2 \left(\frac{a^2 x^3}{3} - \frac{2ax^4}{4} + \frac{x^5}{5} \right)\Big|_0^a \\
&= |A|^2 a^5 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) \\
&= |A|^2 \frac{a^5}{30}.
\end{aligned}
$$

This means that we should take

$$A = \sqrt{\frac{30}{a^5}}.$$

Note that we could have multiplied this by a phase factor, but we chose to have a real value for A.

(b) The expectation value of p is given by the following:

$$\langle p \rangle = \int_0^a \Psi^*(x,0)\hat{p}\Psi(x,0)dx.$$

This is a case where we can jump to the answer. The wavefunction itself is even with respect to the center of the well. One derivative of this even function produces an odd function. So the overall integrand is the product of an even function with an odd function, which is an odd function. Any odd function integrated over a symmetric interval is zero. So

$$\langle p \rangle = 0.$$

To find $\langle p^2 \rangle$ we have to compute

$$\langle p^2 \rangle = \int_0^a \Psi^*(x,0)\hat{p}\hat{p}\Psi(x,0)dx.$$

As part of this consider the action of $\hat{p}^2 = -\hbar^2 \partial^2/\partial x^2$. When acted on the wavefunction, we find

$$\hat{p}^2\Psi(x,0) = -\hbar^2\frac{\partial^2}{\partial x^2}A(ax - x^2) = -\hbar^2 A(-2) = 2A\hbar^2.$$

So our expectation value is just

$$
\begin{aligned}
\langle p^2 \rangle &= |A|^2 \int_0^a (ax - x^2)\cdot 2\hbar^2 dx \\
&= 2\hbar^2\frac{30}{a^5}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right)\Big|_0^a \\
&= 2\hbar^2\frac{30}{a^5}a^3\left(\frac{1}{2} - \frac{1}{3}\right) \\
&= 2\hbar^2\frac{30}{a^2}\frac{1}{6} \\
&= \frac{10\hbar^2}{a^2}.
\end{aligned}
$$

(c) The dispersion in x is given by

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{2a^2}{7} - \frac{a^2}{4}} = a\sqrt{2/7 - 1/4} = a\sqrt{1/28}.$$

A similar calculation for p gives

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{10\hbar^2/a^2} = \frac{\hbar}{a}\sqrt{10}.$$

Their product is

$$\sigma_x\sigma_p = \frac{a}{\sqrt{28}}\frac{\hbar}{a}\sqrt{10} = \frac{\hbar}{2}\sqrt{\frac{10}{7}}$$

So the uncertainty principle is satisfied because $\sqrt{10/7} \geq 1$.

(d) To find the coefficients c_n one need only take the integral of the nth energy eigenfunction against the initial conditions:

$$c_n = \langle \psi_n \, | \, \Psi \rangle = \int_0^a \psi_n^*(x) \Psi(x,0) dx.$$

We can be more specific by plugging in the two functions

$$c_n = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \sqrt{\frac{30}{a^5}} (ax - x^2) dx.$$

We can collect the constants out front and find

$$c_n = \frac{\sqrt{60}}{a^3} \int_0^a (ax - x^2) \sin\left(\frac{n\pi x}{a}\right) dx.$$

There are a fair number of these that will be zero. The wavefunction is symmetric with respect to the center of the well. So as long as the sine term is also symmetric, it is possible to have a non-zero c_n. It is only for odd n that the sine term is also symmetric with respect to the center of the well. The even n's are antisymmetric with respect to the center of the well. So the c_n's for even n are all zero.

2   Energy Measurements

A wave function for a particle is written as the following superposition of stationary states:

$$\Psi(x,0) = \frac{1}{2}\beta_1(x) + \frac{(1-i)}{2}\beta_2(x) + c_3\beta_3(x)$$

where c_3 is some complex constant, $\beta_1(x)$, $\beta_2(x)$ and $\beta_3(x)$ are mutually orthogonal, normalized, energy eigenstates with energies $E_1 = 1$ eV, $E_2 = 2$ eV and $E_3 = 3$ eV.

(a) Determine c_3. Is there any ambiguity in your result? Explain briefly.

(b) Determine the expectation value of the total energy *as a function of time*. Explain briefly.

(c) What is the probability that a single measurement of the energy will result in the expectation value of the energy for this particular wavefunction? Explain briefly.

(d) How would you experimentally determine the expectation value of the energy for this state?

Solution

(a) There are two ways to think about this part of the problem. We can utilize the fact that we know that the sum of the absolute magnitude squared of the coefficients in the expansion has to be 1, or we could apply the normalization condition to the wavefunction directly.

The first method would require that

$$1 = \left|\frac{1}{2}\right|^2 + \left|\frac{1-i}{2}\right|^2 + |c_3|^2.$$

Computing these magnitudes we have

$$1 = \frac{1}{4} + \frac{1-i}{2}\frac{1+i}{2} + |c_3|^2.$$

Some rearrangement gives

$$|c_3|^2 = 1 - \frac{1}{4} - \frac{2}{4} = \frac{1}{4}$$

so we mist have

$$c_3 = \frac{1}{2}.$$

There is some freedom in this result as we could easily well have taken

$$c_3' = \frac{1}{2}e^{i\theta}$$

for some angle θ. The absolute magnitude squared of this c_3' is the same as that for c_3. So the normalization condition will be satisfied equally well in either case.

In the second form, we take the integral of this functions complex conjugate with itself:

$$\int \Psi^*(x,0)\Psi(x,0)dx = \int \left(\frac{1}{2}\beta_1(x) + \frac{(1-i)}{2}\beta_2(x) + c_3\beta_3(x)\right)^* \left(\frac{1}{2}\beta_1(x) + \frac{(1-i)}{2}\beta_2(x) + c_3\beta_3(x)\right)dx$$

There are a great many simplifications we can make here. The left hand side is 1, because we want to enforce normalization. The nine terms on the right will end up expressing either the normalization of the β functions or the orthogonality of these functions:

$$
\begin{aligned}
1 = \frac{1}{4}\int \beta_1^*(x)\beta_1(x)dx \quad & + \quad \frac{(1+i)(1-i)}{4}\int \beta_2^*(x)\beta_2(x)dx + |c_3|^2 \int \beta_3^*(x)\beta_3(x)dx \\
& + \quad \frac{1-i}{4}\int \beta_1^*(x)\beta_2(x)dx + \frac{1+i}{4}\int \beta_2^*(x)\beta_1(x)dx \\
& + \quad \frac{c_3}{2}\int \beta_1^*(x)\beta_3(x)dx + \frac{c_3^*}{2}\int \beta_3^*(x)\beta_1(x)dx \\
& + \quad c_3\frac{1+i}{2}\int \beta_2^*(x)\beta_3(x)dx + c_3^*\frac{1-i}{2}\int \beta_3^*(x)\beta_2(x)dx.
\end{aligned}
$$

The terms in the first row all can be done when we realize that the eigenfunctions are normalized. The six terms in the subsequent rows all are zero because these functions are mutually orthogonal. So we are left with

$$1 = \frac{1}{4} + \frac{2}{4} + |c_3|^2.$$

The solution of this is for

$$c_3 = \frac{1}{2}$$

where we have the same ambiguity as in the other method.

As I said, either of these methods would be a valid solution to this part of the problem.

(b) Again, there are two ways to approach this problem (at least). The first is to find the time-dependent wavefunction given these initial conditions and then take the expectation value directly. The second is to recognize that when have a wavefunction expanded in energy eigenfunctions, that the expectation value of the energy is related to the energy eigenvalues and the coefficients in the expansion. Again, either way would have sufficed for this problem, but in doing the first method, we will derive the result in the second.

We know that the β functions satisfy the time-independent Schrödinger equation,

$$\hat{H}\beta_i(x) = E_i\beta_i(x)$$

where $\hat{H}$ is the Hamiltonian for this system, β_i are the energy eigenfunctions, and the E_i are the associated energy eigenvalues. To make a time-dependent solution of the Schrödinger Equation we have only to tack on the time dependent phase factor:

$$B_i(x,t) = \beta_i(x)e^{-iE_it/\hbar}.$$

The same is true when the initial conditions are a sum of energy eigenfunctions because the Schrödinger Equation is linear. So the initial conditions for this problem can become time dependent by adding the exponential factors appropriately:

$$\Psi(x,t) = \frac{1}{2}\beta_1(x)e^{-iE_1t/\hbar} + \frac{1-i}{2}\beta_2(x)e^{-iE_2t/\hbar} + \frac{1}{2}\beta_3(x)e^{-iE_3t/\hbar}.$$

Next we apply the definition of the expectation value of the energy:

$$\langle H \rangle = \int \Psi^*(x,t)\hat{H}\Psi(x,t)dx.$$

However, we can simplify our life by realizing that our wavefunction is written as a sum of the energy eigenfunctions. So we have

$$\hat{H}\Psi(x,t) = \hat{H}\sum_i c_i\beta_i(x)e^{-iE_it/\hbar} = \sum_i c_iE_i\beta_i(x)e^{-iE_it/\hbar}.$$

Now, when we multiply this by Ψ^*, and integrate, we will find again nine terms: three of which are products of an energy eigenfunction with itself, and six of which are cross terms. In each of the terms, the time dependence drops out, because each copy of $\exp(-iE_it/\hbar)$ is multiplied by $\exp(iE_it/\hbar)$, which is 1. The cross terms cancel because of the orthogonality so we are left with

$$\langle H \rangle = \sum_i |c_i|^2 E_i \int \beta_i^*(x)\beta_i(x)dx.$$

In the previous we have pulled the constants out of the integral. What is left is the normalization condition for the energy eigenfunctions.

$$\langle H \rangle = \sum_i |c_i|^2 E_i$$

This is an equally valid starting location. Notice that there is no time dependence to the expectation value of the energy. This is because we are working with stationary states.

All that is left is to plug in our particular values and find the expectation value:

$$\langle H \rangle = \left|\frac{1}{2}\right| 1 \text{ eV} + \left|\frac{1-i}{2}\right| 2 \text{ eV} + \left|\frac{1}{2}\right| 3 \text{ eV} = \frac{1 \text{ eV}}{4} + \frac{2 \text{ eV}}{2} + \frac{3 \text{ eV}}{4} = 2 \text{ eV}.$$

Again, the reason this expectation value is not time dependent is because the energy eigenfunctions are able to easily furnish a solution of the full Schrödinger Equation. These solutions, which are orthogonal and normalized, cannot give time dependent energy expectation values because the time dependence always meets its conjugate and becomes a factor of 1. This is why these are called stationary states: once they are in such a state, they stay there (up to a phase factor).

(c) Normally you would not expect the expectation value of the energy to be equal to one of the possible values you might get upon measuring the energy. The possible results of a measurement of energy are the energy eigenvalues. In this case, we *do* have an energy eigenvalue equal to 2 eV.

From postulate three, we know that the probability of getting this result is given by the square of the expansion coefficient: $P(E_i) = |c_i|^2 = |\langle \beta_i|\Psi\rangle|^2$. In this case, we are looking for the case when $i = 2$, so we take either

$$P(2 \text{ eV}) = \left|\frac{1-i}{2}\right|^2 = \frac{1}{2}$$

or we take the norm squared of

$$\langle \beta_2 | \Psi \rangle = \int \beta_2^*(x)\Psi(x,0)dx = \sum_{i=1}^3 c_i \int \beta_2^*(x)\beta_i(x)dx = \sum_{i=1}^3 c_i\delta_{2,i} = c_2 = \frac{1-i}{2}$$

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which is the same as the previous probability.

Again, this is not generally possible for a given system and given initial conditions. The expectation value of the energy could be a value of energy that would never be the result of a measurement.

(d) To measure the expectation value of the energy, we would prepare many copies of the system in the prescribed initial conditions. Then, the energy of each copy of the system is measured, once for each copy of the system. We can then take the average of the measured values on this collection of systems, and that would be the expectation value of the energy.