Print your name: Caleb Boyd

Signature: Caleb Boyd

SID [redacted]

Discussion Section: 206

Math 54 Second Midterm Fall 2012 Instructor: D.-V. Voiculescu

This is a "closed book" exam, so you may not bring in or use notes or the textbook. Calculators are not allowed.

Please write your name , SID and Discussion Section # on everything you hand in, including this sheet of paper on which you have to provide the answer to Problem II (the true or false questions). For Problem I you must show the method and calculations you use to get the answers (write the solutions to the questions in Problem I in your blue book). The Requirement is 20 points.

Problem I (3+2+2+3+3 pts) Let A and B be the matrices:

$$A=\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B=\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

a) Find the eigenvalues and eigenvectors of A.

b) Find an invertible matrix S and a diagonal matrix D so that $D = S^{-1} A S$.

c) Apply Gram-Schmidt to the columns of B to find an orthonormal basis of Col (B).

d) Find the 4x4 matrix of the orthogonal projection onto Col (B).

e) Find a least squares solution of $Bx = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Problem II (7 pts, each question 1 pt). Check True or False .

	True	False	
a) If a, b, c, d $\in \mathbb{R}$ then $\sqrt{a^2+4b^2}\sqrt{c^2+d^2} > \lvert ac + 2bd \rvert$	✓		
b) The inverse of an orthogonal matrix is an orthogonal matrix	✓		
c) $\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is an inner product on $\mathbb{R}^2$		✓	
d) If E, F are symmetric 2x2 matrices, then so is EF.		✓	False
e) If W is a subspace of $\mathbb{R}^4$ and {x, y} and {z, t} are bases of W and $W^{\perp}$, then {x, y, z, t} is a basis of $\mathbb{R}^4$.	✓		
f) An orthogonal 2x2 matrix is always diagonalizable		✓	False
g) If M is a 3x2 matrix then rank (A) + nullity (A^T) = 3	✓		

7

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Problem 1

a) $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ triangular, eigenvalues : $\lambda = -1, 1$

3|3

For eigenvectors : $(A - \lambda I)x = 0$

$\lambda = -1$

$(A + I)x = 0$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_2 = 0$, x_1 is free, basis $= x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 1$

$(A - I)x = 0$

$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} -2x_1 + x_2 = 0 \\ x_2 = 2x_1 \\ \frac{x_2}{2} = x_1 \end{cases}$ $\Rightarrow x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

x_2 is free basis $\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

eigenvectors for $\lambda = -1 \Rightarrow x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x \in \mathbb{R}$

eigenvectors for $\lambda = 1 \Rightarrow \cancel{x} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $y \in \mathbb{R}$

b) D will have eigenvalues on diagonal

S will be eigenvectors of those eigenvalues

2|2

$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

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c) $B = \begin{array}{cc} b_1 & b_2 \\ \end{array}$
$$B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

2/2

$$v_1 = b_1$$
$$v_2 = b_2 - \frac{b_2 \cdot b_1}{b_1 \cdot b_1} b_1$$
$$b_2 \cdot b_1 = -1 + 0 - 1 + 0 = -2$$
$$b_1 \cdot b_1 = 1 + 0 + 1 + 0 = 2$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{-2}{2}\right) \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$[1\ 2]\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\cancel{y^T b}$

$y^T b$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

orthogonal basis = v_1, v_2

orthonormal basis = $\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}$

$$\|v_1\| = \sqrt{(-1)^2 + 0^2 + (-1)^2 + 0^2} = \sqrt{1+1} = \sqrt{2}$$
$$\|v_2\| = \sqrt{0^2 + 1^2 + 0^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

orthonormal basis:
$$\left\{ \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cancel{[1\ 2]}$$

d) $y = \hat{y} + z$

orthogonal projection → $\hat{y} = \frac{y \cdot b_1}{b_1 \cdot b_1} b_1 + \frac{y \cdot b_2}{b_2 \cdot b_2} b_2 = \frac{y\, b_1 b_1^T}{b_1^T b_1} + \frac{y\, b_2 b_2^T}{b_2^T b_2} = y\left(\frac{b_1 b_1^T}{b_1^T b_1} + \frac{b_2 b_2^T}{b_2^T b_2}\right)$

$\cancel{\frac{y^T b_1 b_1}{b_1^T b_1}}$

use orthonormal basis of Col(B) from c) as

b_1, b_2 $\quad b_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \; b_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

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d) $b_1^T b_1 = b_1 \cdot b_1 = 1$

3/3

$b_2^T b_2 = b_2 \cdot b_2 = 1$

$$b_1 b_1^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b_2 b_2^T = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$b_1 b_1^T + b_2 b_2^T = \boxed{\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}}$$

e) $Bx = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad B^T B \hat{x} = B^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

3/3

$$B^T B = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+1+0 & -1+0+-1+0 \\ -1+0+-1+0 & 1+1+1+1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$B^T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0+1+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$x_1 - x_2 = 0 \qquad x_1 = x_2$

$2x_2 = 1 \qquad x_2 = \frac{1}{2}$

$$\boxed{\hat{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}$$