<u>Question 1 (10 Points)</u> 1D Collision and Impulse

A uniform rope of mass m *per unit length* hangs vertically from a hook so that the lower end of the rope just touches the horizontal table as shown in Fig. 1(a). The rope is then released from the hook and descends onto the table. The rope is assumed to be completely flexible and when it reaches the table it stops instantaneously. When a length y of the rope has fallen (see Fig. 1 (b)) what is the force exerted by the rope on the table in terms of y, m and g?

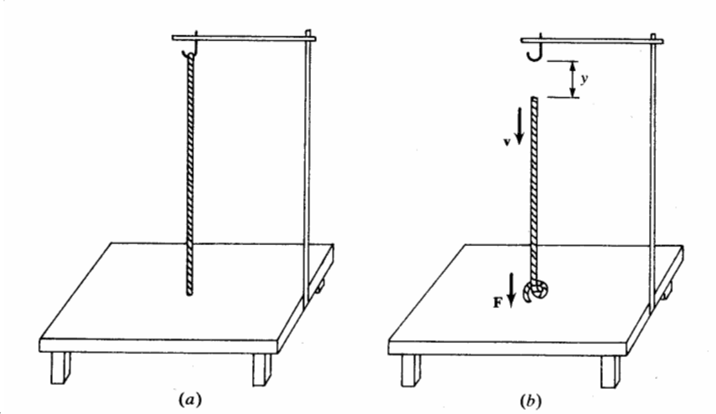


Fig. 1

Solution

The rope descends via free fall onto the table. As it starts from zero initial velocity its velocity v is related to the height y it has fallen by: $v(y)=[2gy]^{1/2}$. The length of the rope which lands on the table during the time dt at that instant is: v(y)dt and the corresponding mass is mvdt

Based on the assumption that the rope stops instantaneously after it lands on the table, the rate at which the rope transfers momentum to the table is:

$dp/dt=[mvdt]v/dt=mv^2=m(2gy)$. According to Newton's equation of motion this is the force exerted on the table by the falling rope. However, there is also rope of length my already on the table and the force they exert due to gravity on the table is myg.

<u>Thus the total force exerted by the rope on the table is 2mgy+mgy=3mgy.</u>

<u>Question 2 (10 Points)</u> Energy & Work

Figure 2 shows a massive spring whose total mass is M_s and whose length when unstretched is L. A smaller mass m is attached to the end of the spring. The mass m and the spring both move on a *frictionless* table. The spring, when stretched, behaves like a rod with Young's modulus E. When the end of the spring is stretched by an amount D the extension of a small element of the spring around the point x can be assumed to be given by (x/L)D. Similarly when the mass m at the end of the spring is moving with velocity V a point x along the spring can be assumed to move with velocity (x/L)V.

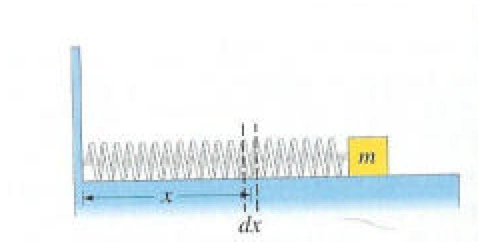


Fig.2

(a) Calculate the potential energy of the *mass m plus the spring* when m is displaced by D.
(b) Calculate the kinetic energy of the mass *m plus the spring* when m is moving with V.
(c) What is the frequency of oscillation of m?

Solution

(a) For a rod of length L, area of cross section A and Young's modulus E the stress (=Force F/Area A)=E(strain) where strain is defined as $\Delta L/L$. Consider the point x along the spring. When it is extended by amount Δx the restoring force on it is: $F=EA(\Delta x/x)$. Δx is given by: $\Delta x = (x/L)D$ so : $F=EA(D/L)$. The direction of F is in the negative x direction. Notice that this force F is independent of x just like the tension in a string is the same everywhere. Thus the spring constant k of the massive spring is =EA/L. The work done by external force in stretching the end of the spring by the amount D is then given by the integral of F d*l* (where d*l* is the displacement of the end of the spring) from *l*=0 to D, just like a regular massless spring. Thus the potential energy stored in the spring is : $(1/2)kD^2$ or <u>$(1/2)(EA/L)D^2$.</u>

(b) Let μ=mass/length of spring. The mass of a small segment of spring of length dx is μ dx. Its velocity $v(x)=d[\Delta x]/dt=(x/L)(dD/dt)$. Let V=velocity of the mass m then $V=dD/dt$. The kinetic energy dK of the small segment of the spring is $dK=(1/2)(\mu dx)[(x/L)V]^2$. The total kinetic energy K of the spring is obtained by integrating dx from x=0 to L or:

$K = \int_0^L \left(\frac{mV^2}{2L^2}\right) x^2 dx$. After integration the result is $K = \left(\frac{mV^2}{2L^2}\right)\left(\frac{L^3}{3}\right)$. Let M_s=mass of spring=μL then $K=(1/2)(M_s/3)V^2$. If we include the kinetic energy of the mass $m=(1/2)mV^2$ we get: <u>total KE of spring and mass=$(1/2)[m+(M_s/3)]V^2$.</u>

(c) The spring has force constant k=(EA/L) and the spring's KE suggests that we can treat it as if its mass is zero but the mass attached to its end is effectively: $m+M_s/3$. The frequency of oscillation of this SHO is therefore: <u>$f=(1/2\pi)[k/(m+M_s/3)]^{1/2}$.</u>

Question 3 (10 Points) Rotational Kinematics

A hockey puck (approximated by a point mass m) slides on a frictionless horizontal ice sheet while tied by a massless string to a vertical post as shown in Fig. 3. The radius of the post is R. Initially the puck has a velocity v_o in the direction perpendicular to the taut string and the length of string is s_o. As the puck continues to move it becomes wrapped around the post shortening the length of the string and hence its distance to the center of the post.

(a) Given the above initial conditions, is it possible for the puck to move in another direction other than perpendicular to the string?

(b) When the puck rotates around the post, will its angular momentum be conserved? Justify your answer.

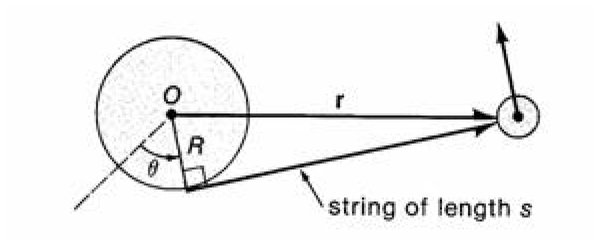


Fig. 3

(c) Calculate the angular velocity of the puck relative to the point O in terms of s, v_o and R.

(d) Derive an expression for the time t for the string to reach the length s from the initial length s_o. In the case your answer involves an integral you do not have to evaluate this integral. However, make sure that you put in the limits of the integration.

Solution

(a) No. The velocity of the puck cannot have any component along the string. Any velocity towards the end of the string will cause the tension to disappear. Any velocity away from the end of the string along the string will mean that the string is stretched.

(b) Although the puck is rotating around the post O but the force acting on it via the string is not parallel to the line joining the center O to the puck so the torque exerted by the string is not zero. As a result the angular momentum is not conserved.

(c) Let the angular velocity of the puck around O be ω. Then ω is given by the *tangential* velocity of the puck relative to O divided by r or

$\omega = v_o \cos\alpha / r$ where α= angle between the string and the line joining O with the puck. From Pythagoras Theorem $\cos\alpha = s/r$ and $r = [R^2+s^2]^{1/2}$ so $\omega = v_o s/[R^2+s^2]$.

(d) Suppose the initial length of the string is s_o and after time t the length of the string becomes s. Suppose we measure the angle θ shown in the figure from the time t=0 then $s = s_o - R\theta$. Thus $ds/dt = -d\theta/dt = -\omega$, the angular velocity of the puck around O.

Using the result obtained in (c) we obtain:

$\omega = v_o s/[R^2+s^2] = -(1/R)(ds/dt)$.

Rewriting this differential equation as :

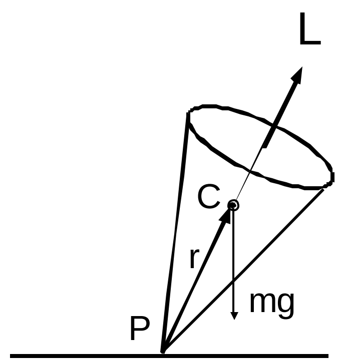
$\int_0^t dt = \int_{s_o}^{s} -\frac{1}{Rv_o}\frac{s^2+R^2}{s}ds$. We find that $t = \int_{s_o}^{s} -\frac{1}{Rv_o}\frac{s^2+R^2}{s}ds$

NOTE: the integral gives the analytic answer: $t = \frac{R}{v_o}\left[\frac{s_o^2 - s^2}{2R^2} + \ln\left(\frac{s_o}{s}\right)\right]$

<u>Question 4 (10 Points)</u> Rotation dynamics

A toy top of mass 220 gm is made to rotate at 15 rev/s with its axis inclined at an angle of 30^o to the vertical. The top *precesses* about the vertical axis at the rate of 1 revolution per 8 seconds. The Center of Mass of the top is determined to lie along its axis of symmetry and its distance from the *tip* of the top is 3.5 cm. What is the moment of inertia of the top? In case you copy an equation from your cheat sheet to solve this problem, you should justify the use of this equation.

Solution



The force due to gravity on the top along the vertical direction produces a torque t on the top. The magnitude of τ is $mgr\sin\alpha$ (where m is the mass of the top, g is the acceleration due to gravity, r is the distance from C, the CM of the top, to its tip P and α is the angle between the axis of the top and the vertical axis and is equal to 30^o in this problem) while the direction of the torque t is pointing into the paper. The torque will change the angular momentum **L(=**Iw where I is the moment of inertia of the top about its axis and ω is the angular velocity of rotation of the top about its axis) of the top so that D**L**=tΔt. Because D**L** is always perpendicular to **L**, the angular momentum of the top will rotate around the x-axis and the velocity of this precession is given by:

|W|=(|D**L**/Δt|)/(|**L**|sinα)=τ/(|**L**|sinα). Thus

Ω=mgr/L or I=mgr/$\omega\Omega$.

Given that the angular velocity ω=14x2π rad/s; the mass m of the top is 0.22 Kg and W =2π/8 rad/s , r=3.5 cm=0.035 m , one obtains :

$I=1.0x10^{-3}$ $Kg.m^2$.

Question 5 (10 Points) Gravitational Force

Assume that the earth's crust is approximately 30 Km deep and has a uniform density of 2.72 gm/cm^3. A geologist tries to measure the fractional change in the acceleration due to gravity (that is $\Delta g/g$) as a way to determine whether there are mineral deposits or natural gases under ground.
(a) Suppose a spherical deposit of pure iron of diameter 5.00 Km and density 7.86 gm/cm^3 lies just beneath the earth's surface. What will be the value of $\Delta g/g$ measured by the geologist immediately above this iron deposit?
(b) Suppose a spherical cavity containing natural gas (which has almost negligible density) lies just beneath the Earth's surface and the geologist measured a value of $\Delta g/g= -1.94x10^{-4}$ immediately above this cavity, what is the diameter of this cavity?

Useful Constant:
Earth's acceleration: $g=9.80\ m/s^2$.
Gravitational Constant $G=6.67x10^{-11}\ Nm^2/Kg^2$.
Mass of Earth=$6.0x10^{24}$ Kg; Radius of Earth=$6.4x10^3$ Km

Solution
(a) The change in the earth's gravity on the earth's surface immediately above the spherical iron deposit due to the presence of the iron deposit can be calculated by replacing the density of the crust with the density of the iron deposit in the Newton's Law of Gravitation: $\Delta g=G(4\pi r^3/3)(\rho_{iron}-\rho_{crust})/r^2$ where G is the universal gravitation constant, ρ_{iron} and ρ_{crust} are, respectively the iron and crust densities and r is the radius of the deposit.
With this formula we can write $\Delta g/g=$ $G(4\pi rR_E^2/3M_E)(\rho_{iron}-\rho_{crust})=$ $3.67x10^{-4}$. (M_E and R_E being the mass and radius of the earth respectively).
(b) From the above equation of $\Delta g/g$ one can calculate the radius R of the cavity knowing the density difference. D=2R=diameter of cavity= 5 Km.

Question 6 (10 Points) Coupled SHO

Two square blocks 1 and 2 have masses m_1 and m_2 respectively. They are connected by 3 springs of identical spring constant k as shown in Fig.6. The masses are sliding on a frictionless surface.

(a) What are the equations governing the displacements Δx_i (where i=1,2) for the two masses?

(b) What are the possible frequencies of vibration of these masses? (Hint: assume that the solutions for $\Delta x_i = A_i \cos\omega t$ where i=1,2)

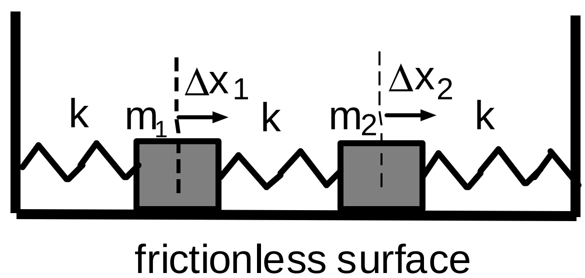


Fig. 6

Solution

(a) Let the force on m_1 be F_1. The equation of motion for m_1 is : $F_1 = (-k\Delta x_1) - k(\Delta x_1 - \Delta x_2) = m_1[d^2\Delta x_1/dt^2]$. Similarly the equation of motion for m_2 is: $F_2 = (-k\Delta x_2) - k(\Delta x_2 - \Delta x_1) = m_2\ [d^2\Delta x_2/dt^2]$.

(b) Assume solutions of the form: $\Delta x_1 = A_1\cos\omega t$ and $\Delta x_2 = A_2\cos\omega t$ and substitute into the above two equations of motion. The resultant linear equations are:

$(m_1\omega^2 - 2k)A_1 + kA_2 = 0$ and $kA_1 + (m_2\omega^2 - 2k)\ A_2 = 0$

These two equations can be combined as: $A_1/A_2 = (-k)/[(m_1\omega^2 - 2k)] = [\ (m_2\omega^2 - 2k)]/(-k)$

Thus the solutions we have written down are valid only when:

$[(m_1\omega^2 - 2k)][\ (m_2\omega^2 - 2k)] - k^2 = 0$. This is an quadratic equation in (ω^2):

$m_1 m_2 w^4 - 2k(m_1 + m_2)w^2 + 3k^2 = 0$ whose solutions are:

$$w^2 = (k/m_1 m_2)\left[(m_1 + m_2) \pm \sqrt{m_1^{\,2} + m_2^{\,2} - m_1 m_2}\right].$$

The acceptable solutions for ω are therefore the positive roots of the above equation :

$$w = (k/m_1 m_2)^{1/2}\left[(m_1 + m_2) \pm \sqrt{m_1^{\,2} + m_2^{\,2} - m_1 m_2}\right]^{1/2}$$

The values of A_1/A_2 can be obtained by substituting these solutions back into the equation: $A_1/A_2 = (-k)/[(m_1\omega^2 - 2k)]$. It turns out that for the higher frequency oscillation the two masses will be oscillating out of phase (ie the two masses are moving in opposite directions) while the in phase oscillation has a lower frequency.

Question 7 (10 Points) Damped SHO

A tuning fork F will oscillate at the frequency of 1.000 KHz when struck and it will oscillate essentially with no damping. Another *identical tuning fork F'* has a rubber band attached to its vibrating arms so that its oscillation is slightly damped. As a result of this damping the frequency of oscillation of F' is also changed slightly. When both tuning forks F and F' are struck simultaneously a beat of frequency of 2 Hz is measured.

(a) what is the frequency of oscillation of F'?

(b) When F' is struck alone, how long will it take for its amplitude to decrease by a factor of 2?

Solution

(a) For two oscillators with frequencies f_1 and f_2 the beat frequency is given by $|f_1-f_2|$ so if f_1=1.000KHz and the beat frequency is 2 Hz, then $f_2=f_1$-2 Hz= 0.998 KHz. Since the second tuning fork has rubber band attached to it has to oscillate at a *lower* frequency.

(b) When the tuning fork F' is struck alone, its amplitude A will decrease exponentially with time t as exp[-αt] where the exponent α is related to the damping constant b, the mass of the SHO by the relation: α=b/2m. To determine b/2m we can use the equation for the frequency of the damped oscillator: f_2 which is related to the frequency of the undamped oscillator f_1 by:

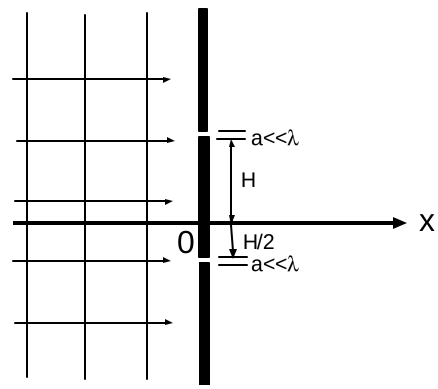
$(2pf_2)^2 = (2pf_1)^2 - \left(\frac{b}{2m}\right)^2$. Thus $a^2 = (2pf_1)^2 - (2pf_2)^2$. When the amplitude decreases by a factor of 2 the time T for this to occur is given by: (1/2)=exp[-αT] or

$T=\log_e 2/\alpha=1.75\times10^{-3}$s.

(a)

Question 8 (10 Points) Wave

Fig.8



A *plane* wave of amplitude A, frequency $f=\omega/2\pi$ and wave length λ is traveling along the x-direction when it hits a wall located at the point x=0 from the left hand side as shown in the Figure 8. There are two identical small holes (size of the holes a are $<<\lambda$) in a wall (the wave cannot penetrate the wall but can be reflected by the wall). One is located at a height H above the x-axis while the other one is located at H/2 below the x-axis. The wave fronts of the incident wave are shown in the figure.

(a) Write down an expression for the magnitude of the displacement D(x,t) of the incident wave as a function of time t and position x.

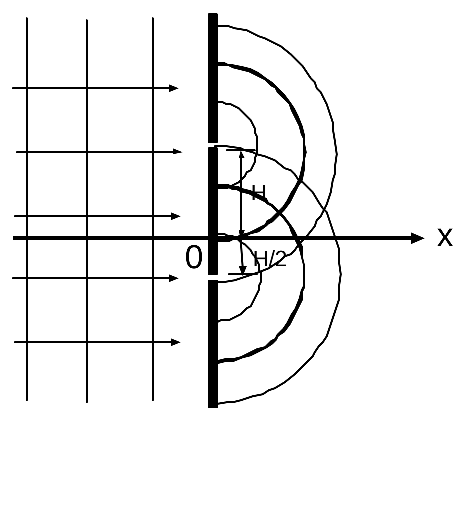
(b) After hitting the wall some of the incident wave are transmitted through the two holes in the wall while the remaining are reflected. Apply Huygen's principle to determine the *shape* of the wave fronts of both the reflected wave and the two waves which escapes through the holes(sketch your result only, do not derive the mathematical expression for the waves).

(c) Derive an equation (ie it is not necessary to solve this equation) from which the locations along the x-axis where the two waves passing through the two small holes will *constructively* interfere can be calculated?

Solution

(a) $D(x,t)=A\cos[\omega t-(2\pi/\lambda)x]$.

(b) From Huygen's principle, the wave fronts at the two small holes will generate spherical wavelets and these will produce the new hemispherical wave fronts as shown in following figure:



(c) Let x be the position of a point along the x axis. The distance traveled by the two spherical waves originating from the holes at height H and H/2 are, respectively,:

$d_1=[H^2+x^2]^{1/2}$ and $d_2=[(H/2)^2+x^2]^{1/2}$. The phase difference between the two spherical waves will be: $\Delta\theta=2\pi[d_1-d_2/\lambda]$. <u>For constructive interference to occur at x we must have : $\Delta\theta$=multiples of 2π. Another to state this is: the path difference is equal to an integral multiple of the wavelength.</u> ?

Question 9 (10 Points) Fluid dynamics

A cylindrical tank has a radius of 9 m and rests on top of a platform 6 m high. A siphon tube is used to drain this tank which is filled with *water* to a height of 3 m as shown in Figure 9. The siphon tube rises to 5 m above the bottom of the tank and then descends to the ground where the water drains out. The siphon tube has a diameter of 2 cm. Assume that the water enters the siphon with almost zero velocity.

(a) Calculate the velocity of the water when it exits the siphon tube.
(b) How long will it take to drain all the water from the tank?

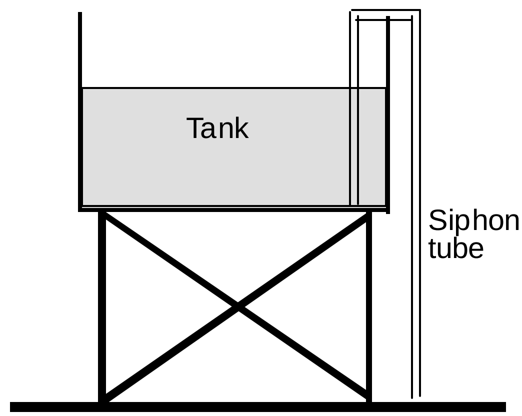


Figure 9

Solution

(a) Apply the Bernouilli equation to these 2 points: 1 ⇔ the entrance to the siphon and 2 ⇔ to the exit point of the siphon:

$$P_1 + r_i g y_1 + \frac{r_1 v_1^2}{2} = P_2 + r_2 g y_2 + \frac{r_2 v_2^2}{2}$$

Since water is assumed to be incompressible, we will write $\rho_1=\rho_2=\rho$. In addition we can assume that $v_1 \sim 0$. The pressure P_1 is then $P_a+\rho gH$ where P_a is atmospheric pressure. Similarly we assume that $P_2=P_a$. If h=height of platform then $y_1-y_2=h$. Putting these results into the Bernouille equations one obtains:

$g(h+H) = \frac{v_2^2}{2}$ or $\underline{v_2=[2g(H+h)]^{1/2}}$.

(a) Let A_1 and A_2 be, respectively, the area of cross section of the tank and of the siphone tube. Continuity Equation then requires that $-\rho A_1(dH/dt)=\rho A_2 v_2$. This differential equation can be expressed as : $-dH/v_2=(A_2/A_1)dt$. When express v_2 in terms of H we obtain: $-\frac{dH}{\sqrt{2g(H+h)}} = \frac{A_2}{A_1}dt$. Let time t=0 be the starting time when water drains from the tank through the siphon and the height of the water in the tank $H=H_o=3$ m. Let T be the time when the tank is empty or H=0. The definite integrals on both side of the above equation becomes: $-\int_{H_o}^{0} \frac{dH}{\sqrt{2g(H+h)}} = \int_0^T \frac{A_2}{A_1}dt$. After integration the result is:

$T = \left(\frac{A_2}{A_1}\right)\sqrt{\frac{2}{g}}\left[\sqrt{H_o+h} - \sqrt{h}\right]$$=(9/0.01)^2[2/9.8]^{1/2}[(9)^{1/2}-(6)^{1/2}]$ s=<u>56 hours.</u>

<u>Question 10 (10 Points)</u> Statics

An uniform rod with mass m=2 Kg is suspended by a rope DC inside a hole. The ends of the rod A and E are in contact with the frictionless wall of the hole. The angle between the rod and the walls is equal to 60^o as shown in Figure 10.

A weight of mass M=5 Kg is suspended from the point B along the rod. Find

(a) the tension in the rope CD and

(b) the forces acting on the ends A and E of the rod due to the walls.

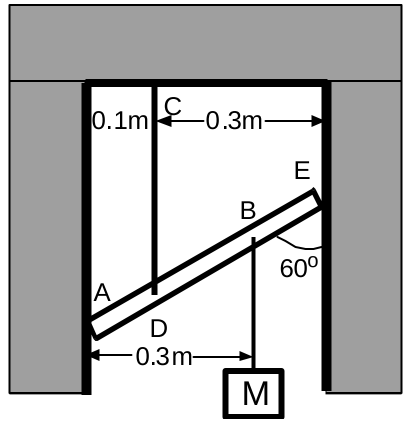
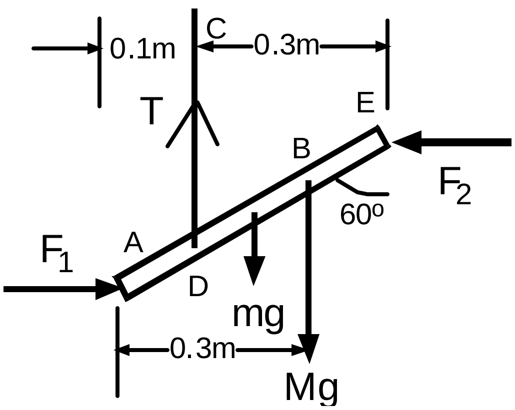


Fig. 10

Solution

First draw the free body diagram for the rod as shown in the following figure.



Notice that the forces F1 and F2 on the ends of the rod are perpendicular to the wall since there is no friction.

(a) Equating all the vertical forces: T=tension in the string=total downward force=(m+M)g=7g N=<u>68.6 N.</u>

(b) Since there is no rotation the net torque on the rod must be zero. If we take the moment of all the forces around the point A we get: $F_2a+T(0.1)=mg(0.2)+Mg(0.3)$ where a is the vertical distance between A and E and is equal to 0.4/[3]1/2 m or <u>F_2=51.1 N.</u>

Since there are only 2 horizontal forces acting on the rod, they must be equal and force for static equilibrium of the rod or <u>**$F_2=-F_1$.**</u>