First Midterm Examination
Closed Books and Closed Notes

Question 1
Vehicle Dynamics (25 Points)

An engineer is commissioned to design a track profile to test the suspension and driving dynamics of a car. The road has the following vertical profile:

$$y = f(x) \text{ where } f(x) = A\sin\left(\frac{\pi x}{L}\right), \tag{1}$$

where A and L are constants. The car is modeled as a particle of mass m which is subject to a normal force $\mathbf{N}$, a traction force $\mathbf{F}_T = F_T\mathbf{e}_t$, a drag force $-mC_d\left\|\mathbf{v}\right\|^2\mathbf{v}$ and a gravitational force $-mg\mathbf{E}_y$.

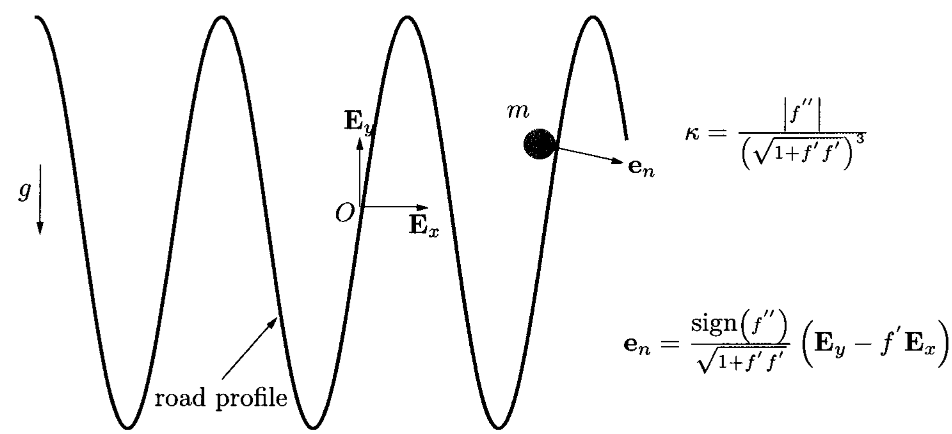


Figure 1: *Schematic of a particle of mass m moving on a road.*

(a) *(8 Points)* Starting from the following representation for the position vector of the particle,

$$\mathbf{r} = x\mathbf{E}_x + f(x)\mathbf{E}_y, \tag{2}$$

derive expressions for the speed v, velocity vector $\mathbf{v}$, and acceleration vector $\mathbf{a}$ of the particle. Assuming $\dot{x} > 0$, for which locations on the road are $v = \dot{x}$?

(b) *(5 Points)* Draw a freebody diagram of the particle.

(c) *(6 Points)* Assuming that the car is moving at a constant speed v_0 on the given road profile with $\dot{x} > 0$, show that the traction force and the normal force acting on the car are, respectively,

$$\mathbf{F}_T = \left(mC_dv_0^3 + mg??\right)\mathbf{e}_t, \qquad \mathbf{N} = \left(mv_0^2\kappa + mg???\right)\mathbf{e}_n. \tag{3}$$

For full credit, you need to give correct expressions for the terms denoted by ?? and ???.

(d) *(6 Points)* Using the results of **(c)**, for which locations on the track are $\mathbf{F}_T\cdot\mathbf{e}_t$ maximized and minimized? Give a physical interpretation of your solutions. Hint: $\frac{d}{dx}\frac{mgf'}{\sqrt{1+f'f'}} = \frac{mgf''}{\left(\sqrt{1+f'f'}\right)^3}$.

Question 2

A Particle in a Rotating Tube (25 POINTS)

As shown in Figure 2, a particle of mass m is attached to a fixed point O by a linearly elastic spring. The spring has a stiffness K and an unstretched length L. The particle is free to move inside a rough tube which is rotating about a vertical axis with a speed $\Omega(t)$.

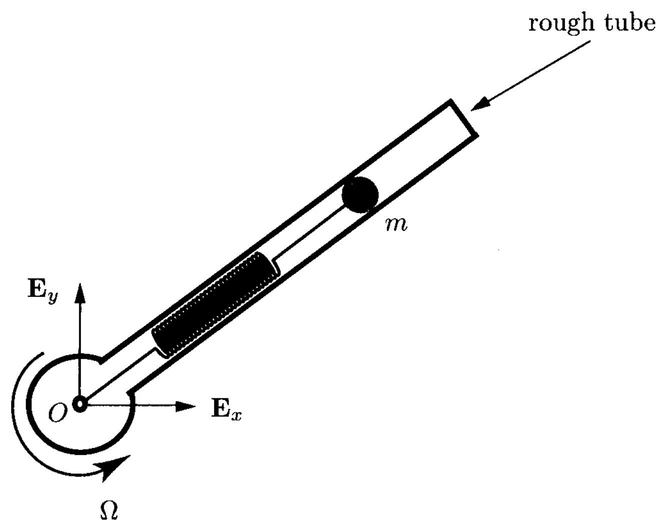


Figure 2: *Schematic of a particle of mass m which is attached to a fixed point O by a linearly elastic spring. A vertical gravitational force $-mg\mathbf{E}_z$ acts on the particle and the particle is free to move on the inside of the rough tube which is rotating about the vertical axis.*

(a) Starting from the standard representation for the position vector

$$\mathbf{r} = r\mathbf{e}_r, \tag{4}$$

establish expressions for the velocity $\mathbf{v}$ and $\mathbf{v}_{\text{rel}}$ and acceleration $\mathbf{a}$ vectors of the particle. In your solution, it is not necessary to derive the intermediate results $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$ and $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$.

(b) Draw a freebody diagram of the particle. Your freebody diagram should include a clear expression for the spring force and accommodate both the static and dynamic friction cases.

(c) Suppose that the particle is moving relative to the tube. Show that the differential equation governing the motion of the particle is

$$m\left(\ddot{r} - r\Omega^2\right) = ?? - \mu_k \|\mathbf{N}\| ???. \tag{5}$$

For full credit show how $\mathbf{N}$ in (5) can be determined and provide expressions the terms denoted by ?? and ???

(d) Suppose that the particle is stationary relative to the tube: $r = r_0$. Show that it is possible for the particle to remain in this state provided

$$\left|\frac{K}{m}(r_0 - L) - r_0\Omega^2\right| \le \mu_s\sqrt{g^2 + r_0^2\dot{\Omega}^2}. \tag{6}$$

(e) Henceforth restrict attention to the case where the tube is smooth and $\Omega = \Omega_0$ is constant. Suppose the particle is placed in the tube with $r = r_0$ and $\dot{r} = 0$. Show that in the subsequent motion

$$\dot{r}^2 = \Omega_0^2\left(r^2 - r_0^2\right) - \frac{K}{m}\left((r - L)^2 - (r_0 - L)^2\right). \tag{7}$$

Hint: one approach to solving this problem is to use the identity $vdv = adr$.

QUESTION 1

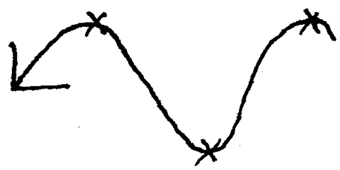
a) $\underline{r} = x\underline{E}_x + f(x)\underline{E}_y$

$\underline{v} = \dot{x}\,(\underline{E}_x + f'\underline{E}_y) \quad \Rightarrow \quad v = |\dot{x}|\sqrt{1+f'f'}$

$\underline{e}_t = \frac{\underline{v}}{v} = \frac{|\dot{x}|}{\dot{x}} \frac{1}{\sqrt{1+f'f'}}(\underline{E}_x + f'\underline{E}_y)$

$\underline{a} = \ddot{x}\,(\underline{E}_x + f'\underline{E}_y) + \dot{x}^2 f''\underline{E}_y$

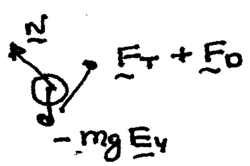
when $\dot{x} > 0$, $\quad v = \dot{x}$ when $f' = 0 \Rightarrow \frac{A\pi}{L}\cos\left(\frac{\pi x}{L}\right) = 0$



$\Rightarrow \frac{x}{L} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc

i.e. at crests and troughs in road.

b)



$\underline{N} = N\underline{e}_n \qquad \underline{F}_T = F_T\underline{e}_t$

$\underline{F}_D = -mc_d v^2 \underline{v} = -mc_d v^3 \underline{e}_t$

c) $\underline{F} = m\underline{a}$ with $\underline{a} = \dot{v}\underline{e}_t + \kappa v^2 \underline{e}_n = \kappa v_0^2 \underline{e}_n$

From $\underline{F} = m\underline{a} \cdot \underline{e}_t = 0 \qquad F_T = mg\underline{E}_y \cdot \underline{e}_t + mc_d v_0^3$

$$= \frac{mg f'}{\sqrt{1+f'f'}} + mc_d v_0^3$$

From $\underline{F} = m\underline{a} \cdot \underline{e}_n \qquad N - mg\underline{E}_y \cdot \underline{e}_n = mv_0^2/\rho = mv_0^2 \kappa$

Hence $\underline{N} = \left(\frac{mg\ \mathrm{sgn}(f'')}{\sqrt{1+f'f'}} + mv_0^2 \kappa\right)\underline{e}_n$

Notes: using f instead of $A\sin\frac{\pi x}{L}$ saves consider amounts of algebra and time in solns.
a common error was to assume $f' = \dot{f}$. This is not true: $\dot{x}f' = \dot{f}$.

(d) $$F_T = \frac{mg f'}{\sqrt{1+f'f'}} + mc_d V_0^3$$

To find locations where F_T is maximized consider

$$\frac{\partial F_T}{\partial x} = 0 \Rightarrow \frac{mg f''}{\sqrt{1+f'f'}} - \frac{mg f'f'f''}{(\sqrt{1+f'f'})^3} = 0$$

$$\Rightarrow \frac{mg f''}{(\sqrt{1+f'f'})^3} = 0$$

Hence F_T is maximized / minimized when $f'' = 0$

$$\Rightarrow \frac{A\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right) = 0$$

Hence $\frac{\pi x}{L} = 0, \pi,$ etc. That is, the points of inflection of $y = f(x)$.

F_T is maximized when $\frac{\pi x}{L} = 0, 2\pi, 4\pi$: $f' = \frac{A\pi}{L}$

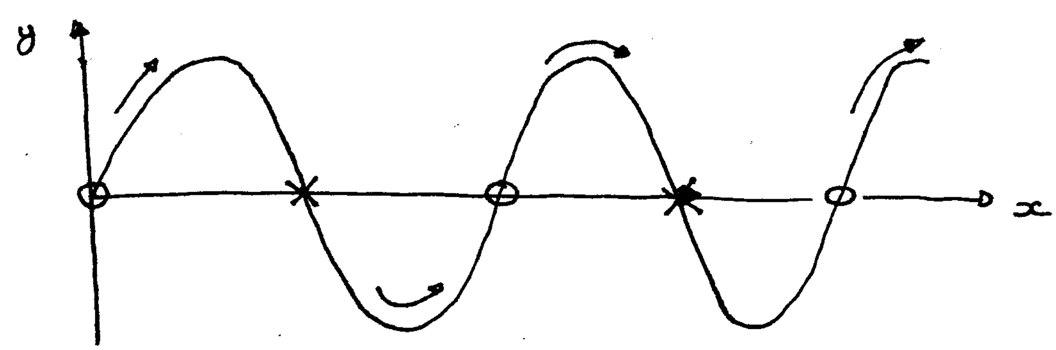
$$(F_T)_{max} = \frac{mg\frac{A\pi}{L}}{\sqrt{1 + \frac{A^2\pi^2}{L^2}}} + mc_d V_0^3$$

i.e. car is moving uphill.

F_T is minimized when $\frac{\pi x}{L} = \pi, 3\pi, 5\pi$: $f' = -\frac{A\pi}{L}$

$$(F_T)_{min} = \frac{-mg\frac{A\pi}{L}}{\sqrt{1 + \frac{A^2\pi^2}{L^2}}} + mc_d V_0^3$$

i.e. car is moving downhill.



O = locations for max F_T , X = locations for min F_T.

QUESTION 2

a) $\underline{r} = r\underline{e}_r$

$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta = \dot{r}\underline{e}_r + r\Omega\underline{e}_\theta$

$\underline{a} = \ddot{r}\underline{e}_r + \dot{r}\Omega\underline{e}_\theta + \dot{r}\dot{\theta}\underline{e}_\theta + r\dot{\Omega}\underline{e}_\theta + r\Omega(-\Omega)\underline{e}_r$

$= (\ddot{r} - r\Omega^2)\underline{e}_r + (r\dot{\Omega} + 2\dot{r}\Omega)\underline{e}_\theta$

$\underline{v}_{rel} = \dot{r}\underline{e}_r$

b)

$\underline{N} = N_\theta\underline{e}_\theta + N_z\underline{E}_z$

$\underline{F}_s + \underline{F}_f$

$-mg\underline{E}_z$

$\underline{F}_f = F_{fr}\underline{e}_r$ (for static friction)

$= -\mu_K \|\underline{N}\| \frac{\dot{r}}{|\dot{r}|}\underline{e}_r$ (for dynamic friction)

$\underline{F}_s = -K(r-L)\underline{e}_r$

c) $\underline{F} = m\underline{a}$

$\cdot\underline{e}_r \quad m(\ddot{r} - r\Omega^2) = -K(r-L) - \mu_K\|\underline{N}\|\frac{\dot{r}}{|\dot{r}|}$

$\underline{N}$ is determined from $(\underline{F} = m\underline{a})\cdot\underline{e}_\theta$ and $(\underline{F} = m\underline{a})\cdot\underline{E}_z$

$N_\theta = m(r\dot{\Omega} + 2\dot{r}\dot{\Omega})$

$N_z = mg$

d) For static friction case $\ddot{r} = \dot{r} = 0$. From $\underline{F} = m\underline{a}$

$\cdot\underline{e}_r \quad F_{fr} = K(r_0 - L) - mr_0\Omega^2$

$\cdot\underline{e}_\theta$ and $\underline{E}_z \quad \underline{N} = mg\underline{E}_z + mr\dot{\Omega}\underline{e}_\theta$

From static friction criterion

$\|\underline{F}_f\| \le \mu_s\|\underline{N}\| \Rightarrow |K(r_0 - L) - mr_0\Omega^2| \le \mu_s\sqrt{m^2g^2 + m^2r^2\dot{\Omega}^2}$

Dividing by m gives desired final form of this inequality.

e) When tube is smooth equation of motion is

$$m(\ddot{r} - r\Omega_0^2) = -K(r-L)$$

Hence $$\ddot{r} = r\Omega_0^2 - \frac{K}{m}(r-L)$$

This is an equation of the form $a = a(r)$ where $a = \ddot{r}$.

Hence integrating $v\,dv = a\,dr$

$$v^2 - v_0^2 = 2\int_{r_0}^{r} u\Omega_0^2 - \frac{K}{m}(u-L)\,du$$

$$= \Omega_0^2(r^2 - r_0^2) - \frac{K}{m}(r-L)^2 + \frac{K}{m}(r_0-L)^2$$

But $v_0 = 0$ $v = \dot{r}$

So

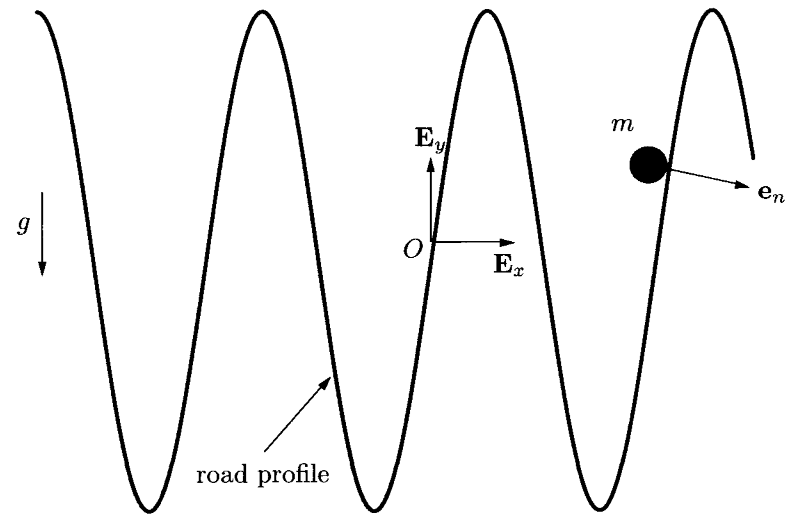
$$\dot{r}^2 = \Omega_0^2(r^2 - r_0^2) - \frac{K}{m}\left((r-L)^2 - (r_0-L)^2\right)$$

This result can also be obtained from the work-energy theorem $\dot{E} = \underline{F}_{nc}\cdot\underline{v}$ because $\underline{F}_{nc}\cdot\underline{v} = 0$ and so E is conserved.

Notes: The most common error for this problem was to assume gravity was in the $-\underline{E}_y$ direction. This error can be noted when trying to solve (d).

Solution to Question 1

Vehicle Dynamics (25 Points)



Q1 :
Q2 :
Tot :

Solution to Question 2

A Particle in a Rotating Tube (25 POINTS)