University of California, Berkeley

Math 1B Midterm 2 Solutions

Slobodan Simić, Spring 2012

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	Score
1	20
2	20
3	20
4	20
5	20
Total	100

Instructions. Read the problems very carefully to be sure you understand the statements. Justify your answers. Show all your work as clearly as possible and *circle* the final answer to each problem. When giving explanations, write complete sentences. If you have any questions, please ask any of the proctors. When you are done with the exam, please hand it to the nearest (not necessarily your own) GSI. If you finish early, please leave quietly.

1. (20 points) For each of the following statements, determine if the conclusion ALWAYS follows from the assumptions, if the conclusion is SOMETIMES true given the assumptions, or if the conclusion is NEVER true given the assumptions. You do not need to show any work or justify your answers to these questions – only your circled answer will be graded.

(a) If $a_n \to 0$, as $n \to \infty$, then $\sum (-1)^n a_n$ is convergent.

ALWAYS $\boxed{SOMETIMES}$ NEVER

(b) If $\sum a_n$ is convergent and $a_n > 0$ for all n, then $\sum \sqrt{a_n}$ is convergent.

ALWAYS $\boxed{SOMETIMES}$ NEVER

(c) If $\sum a_n$ converges, then $(a_1 + a_2 + \cdots + a_n)/n \to 1$, as $n \to \infty$.

ALWAYS SOMETIMES $\boxed{NEVER}$

(d) If (b_n) is convergent sequence, then the series $\sum (b_n - b_{n+1})$ is convergent.

$\boxed{ALWAYS}$ SOMETIMES NEVER

(e) If $\sum (-2)^n c_n$ is divergent, then $\sum 3^n c_n$ is convergent.

ALWAYS SOMETIMES $\boxed{NEVER}$

2. (20 points) Compute the limits of the following sequences:

(a) $a_n = \sqrt{\dfrac{12n^4 + \pi n}{3n^4 - n^2 + 2012}}$ (b) $b_n = \sqrt{n}\arctan\dfrac{\pi}{\sqrt{n}}$

Solution: (a) Dividing the denominator and the numerator inside the square root by n^4, we obtain:

$$\begin{aligned} a_n &= \sqrt{\frac{12 + \dfrac{\pi}{n^3}}{3 - \dfrac{1}{n^2} + \dfrac{2012}{n^4}}} \\ &\to \sqrt{\frac{12}{3}} \\ &= \boxed{2}. \end{aligned}$$

(b) Using the fact that $\arctan x/x \to 1$, as $x \to 0$, we obtain

$$\begin{aligned} b_n &= \frac{\arctan \dfrac{\pi}{\sqrt{n}}}{\dfrac{\pi}{\sqrt{n}}}\pi \\ &\to 1 \cdot \pi \\ &= \boxed{\pi.} \end{aligned}$$

3. (20 points) For which values of p, where $-\infty < p < \infty$, is the series

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{n^p}$$

convergent?

Solution: Let

$$a_n = \frac{\sin\left(\frac{1}{n^2}\right)}{n^p} \qquad \text{and} \qquad b_n = \frac{1}{n^{p+2}}.$$

Using the fact that $\sin x/x \to 1$, as $x \to 0$, we obtain

$$\frac{a_n}{b_n} = \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} \to 1,$$

as $x \to \infty$. By the Limit Comparison Test,

$$\sum a_n \text{ converges iff } \sum b_n \text{ converges.}$$

Since $\sum b_n$ converges iff $p + 2 > 1$, i.e., iff $p > -1$, it follows that

$$\boxed{\sum a_n \text{ converges iff } p > -1.}.$$

4. (20 points) For which values of x does the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{(n+1)\ 3^n}$$

converge?

Solution: Let

$$a_n = \frac{(x-1)^n}{(n+1)\ 3^n}.$$

Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3} \frac{n+1}{n+2} |x-1| \to \frac{|x-1|}{3},$$

as $n \to \infty$. By the Ratio Test, the radius of convergence of the given series is

$$\boxed{R = 3,}$$

and the series converges absolutely on $(-2, 4)$ and it diverges for $|x-1| > 3$. Let us examine the points $x = -2$ and $x = 4$.

If $x = -2$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1},$$

which is convergent, by the Alternating Series Test.

If $x = 4$, the series becomes the harmonic series, $\sum 1/(n+1)$, which is divergent.

To summarize, the given power series converges if and only if

$$\boxed{-2 \leq x < 4}.$$

5. (20 points) Consider the function f defined by

$$f(x) = \begin{cases} \dfrac{e^{x^2} - 1 - x^2}{x^3} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Find the MacLaurin series for f. What is its radius of convergence?

(b) Express

$$\int_0^1 f(x)\,dx.$$

as the sum of an infinite series.

Solution: (a) We will use the MacLaurin series for the exponential function,

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

whose radius of convergence is $R = \infty$. Substituting $t = x^2$, we obtain

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \sum_{n=2}^{\infty} \frac{x^{2n}}{n!}, \tag{1}$$

which also converges for all x. Subtracting $1 + x^2$ and dividing by x^3, for $x \neq 0$, we obtain

$$f(x) = \frac{e^{x^2} - 1 - x^2}{x^3} = \sum_{n=2}^{\infty} \frac{x^{2n-3}}{n!} = \frac{x}{2!} + \frac{x^3}{3!} + \frac{x^5}{4!} + \cdots.$$

It remains to check that the sum of this series equals f also when $x = 0$, but this is clear, since both vanish there. Therefore,

$$\boxed{f(x) = \sum_{n=2}^{\infty} \frac{x^{2n-3}}{n!}.}$$

The radius of convergence of this series is the same as the radius of convergence of the series in (1), i.e., $\boxed{R = \infty.}$.

(b) Since power series can be integrated term by term on their interval of convergence, we obtain:

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 \sum_{n=2}^{\infty} \frac{x^{2n-3}}{n!}\,dx \\ &= \sum_{n=2}^{\infty} \int_0^1 \frac{x^{2n-3}}{n!}\,dx \\ &= \boxed{\sum_{n=2}^{\infty} \frac{1}{n!}\frac{1}{2n-2}.}\end{aligned}$$