University of California, Berkeley

Math 1B Midterm 1 Solutions

Slobodan Simić, Spring 2012

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	Score
1	20
2	20
3	20
4	20
5	20
Total	100

Instructions. Read the problems very carefully to be sure you understand the statements. Justify your answers. Show all your work as clearly as possible and $\boxed{\textit{circle}}$ the final answer to each problem. When giving explanations, write complete sentences. If you have any questions, please ask any of the proctors. When you are done with the exam, please hand it to your GSI. If you finish early, please leave quietly.

1. (20 points) Compute the following integral:

$$\int_0^{\sqrt{3}/2} \arcsin x \, dx.$$

Solution: We will use integration by parts with

$$u = \arcsin x, \qquad dv = dx.$$

Then

$$du = \frac{dx}{\sqrt{1-x^2}}, \qquad v = x,$$

so

$$\begin{aligned} \int_0^{\sqrt{3}/2} \arcsin x \, dx &= x \arcsin x\Big|_0^{\sqrt{3}/2} - \int_0^{\sqrt{3}/2} \frac{x \, dx}{\sqrt{1-x^2}} \\ &= \frac{\sqrt{3}}{2}\frac{\pi}{3} - I. \end{aligned}$$

To compute I, we can make a substitution $w = 1 - x^2$. This yields

$$I = -\sqrt{1-x^2}\Big|_0^{\sqrt{3}/2} = \frac{1}{2}.$$

Therefore,

$$\boxed{\int_0^{\sqrt{3}/2} \arcsin x \, dx = \frac{\pi\sqrt{3}}{6} - \frac{1}{2}.}$$

Alternative solution: Substitution $x = \sin t$, $0 \le t \le \pi/3$ yields

$$\int_0^{\sqrt{3}/2} \arcsin x \, dx = \int_0^{\pi/3} t \cos t \, dt.$$

Integration by parts with $u = t$, $dv = \cos t \, dt$ gives

$$\begin{aligned} \int_0^{\pi/3} t \cos t \, dt &= t \sin t\Big|_0^{\pi/3} - \int_0^{\pi/3} \sin t \, dt \\ &= \boxed{\frac{\pi\sqrt{3}}{6} - \frac{1}{2}}. \end{aligned}$$

2. (20 points) Compute the following two integrals:

(a)

$$\int x \arctan x \, dx.$$

(b)

$$\int_0^{1/\sqrt{2}} \frac{dx}{(1-x^2)^{3/2}}.$$

Solution: (a) We integrate by parts with

$$\begin{aligned} u &= \arctan x, & dv &= x\,dx \\ du &= \frac{dx}{1+x^2}, & v &= \frac{x^2}{2}. \end{aligned}$$

Then:

$$\begin{aligned} \int x \arctan x \, dx &= \frac{x^2}{2} \arctan x - \int \frac{1}{2} \frac{x^2}{1+x^2} \, dx \\ &= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \boxed{\frac{x^2}{2} \arctan x - \frac{1}{2}(x - \arctan x) + C.} \end{aligned}$$

(b) We substitute

$$x = \sin t.$$

Then $dx = \cos t \, dt$ and since $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, the new limits of integration are 0 and $\pi/4$. Thus:

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{dx}{(1-x^2)^{3/2}} &= \int_0^{\pi/4} \frac{\cos t \, dt}{\cos^3 t} \\ &= \int_0^{\pi/4} \frac{dt}{\cos^2 t} \\ &= \tan t \big|_0^{\pi/4} \\ &= \tan \frac{\pi}{4} - \tan 0 \\ &= \boxed{1}. \end{aligned}$$

3. (20 points) Compute the following integral:

$$\int_0^{\pi/2} \sin^5 x \, dx.$$

Solution: We separate a $\sin x$, express the rest in terms of $\cos x$ and substitute

$$u = \cos x.$$

The new limits of integration are 1 and 0 (in that order), and $du = -\sin x \, dx$. It follows that

$$\begin{aligned}
\int_0^{\pi/2} \sin^5 x \, dx &= \int_0^{\pi/2} \sin x (1 - \cos^2 x)^2 \, dx \\
&= \int_0^1 (1 - u^2)^2 \, du \\
&= \int_0^1 (1 - 2u^2 + u^4) \, du \\
&= 1 - \frac{2}{3} + \frac{1}{5} \\
&= \boxed{\frac{8}{15}}.
\end{aligned}$$

Here, we used the following useful formula:

$$\int_0^1 u^k \, du = \frac{1}{k+1},$$

for all $k \neq -1$.

4. (20 points) Determine whether the following integrals converge or diverge. If an integral converges, compute its value.

(a)

$$\int_e^\infty \frac{dx}{x(\ln x)^2}.$$

(b)

$$\int_2^\infty \frac{x^2+1}{\sqrt{x^6-x^4}}dx.$$

Solution: (a) Using a substitution $u = \ln x$, we obtain

$$\begin{aligned}\int_e^\infty \frac{dx}{x(\ln x)^2} &= \lim_{t\to\infty}\int_e^t \frac{dx}{x(\ln x)^2}\\ &\overset{u=\ln x}{=} \lim_{t\to\infty}\int_1^{\ln t}\frac{du}{u^2}\\ &= \lim_{t\to\infty}\left(-\frac{1}{u}\right)\Bigg|_1^{\ln t}\\ &= \lim_{t\to\infty}\left(1-\frac{1}{\ln t}\right)\\ &= 1.\end{aligned}$$

Therefore, the integral **converges**.

(b) Since

$$\frac{x^2+1}{\sqrt{x^6-x^4}} \geq \frac{x^2}{\sqrt{x^6}} = \frac{1}{x},$$

for all $x \geq 2$ and the integral $\int_2^\infty dx/x$ diverges, it follows by Comparison Test that the given integral **diverges**.

5. (20 points) Find the arc length of the curve defined by

$$y = \ln(1 - x^2), \qquad 0 \le x \le \frac{1}{2}.$$

Solution: We have:

$$y' = -\frac{-2x}{1 - x^2},$$

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{4x^2}{(1 - x^2)^2}} = \frac{1 + x^2}{1 - x^2}.$$

Long division of polynomials yields

$$\frac{1 + x^2}{1 - x^2} = -1 + \frac{2}{1 - x^2}.$$

Therefore, the arc length equals

$$\begin{aligned} L &= \int_0^{1/2} \sqrt{1 + (y')^2}\, dx \\ &= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx \\ &= -\frac{1}{2} + \int_0^{1/2} \frac{2}{1 - x^2}\, dx. \end{aligned}$$

To compute the last integral, we use partial fractions. Since $1 - x^2 = (1 + x)(1 - x)$, we obtain

$$\frac{2}{1 - x^2} = \frac{A}{1 + x} + \frac{B}{1 - x}.$$

This implies

$$2 = A(1 - x) + B(1 + x),$$

for all real numbers x. Substituting $x = -1$, we obtain $2 = 2A$, so $A = 1$. Using $x = 1$, we get $2 = 2B$, so $B = 1$. Therefore,

$$\begin{aligned} \int_0^{1/2} \frac{2}{1 - x^2}\, dx &= \int_0^{1/2} \left(\frac{1}{1 - x} + \frac{1}{1 + x}\right) \\ &= \left(\ln|1 + x| - \ln|1 - x|\right)\Big|_0^{1/2} \\ &= \ln\frac{3}{2} - \ln\frac{1}{2} \\ &= \ln 3. \end{aligned}$$

Putting things together, we obtain the final answer:

$$\boxed{L = \ln 3 - \frac{1}{2}.}$$