1 Problem 1

1.1 Part a

We want to find the lengths of the two rods given the initial lengths, a change in temperature, and their coefficients of linear expansion. To do this, we use the formula $\Delta l = l_0 \alpha \Delta T$, so that the final length is given by $l_f = l_0(1 + \alpha \Delta T)$.

If we plug in the numbers, for steel we have an initial length of 98 cm, an expansion coefficient of $12 \times 10^{-6}/°\text{C}$, with a temperature change of 1000°C - 200°C = 800°C. This gives a final length of 98.94 cm (once rounded). For quartz, the temperature change is the same, the initial length is 100 cm, and α is $0.4 \times 10^{-6}/°\text{C}$. This gives a final length of 100.032 cm.

1.2 Part a rubric

Part a was graded out of 8 points, with 4 points given for each calculation. These broke down to 3 for the formula and 1 for plugging in numbers. If there were mistakes with numbers for both calculations, it was -2 total, but mistakes in the formula were deducted from the total, rather than by problem. The amount deducted was greatest for mistakes with the wrong units, such as $l = l_0 + \alpha \Delta T$, since the second term does not have units of length. Large deductions were also given for saying that the final length of the rod was $l_0 \alpha \Delta T$, which is the change in the length. A small deduction (1 pt total) was given for providing only the change in length, and not the total final length, if the result was correctly labeled as the change.

1.3 Part b

This part asks whether the two solid rods can ever be the same length. There are actually several possible solutions. The following explanations are (much) longer than they would need to be on the exam, just to ensure that the reasoning is clear.

The easiest solution is as follows: The first step is to note that the steel starts off shorter than the quartz and also changes length more readily with change in temperature, so that the steel will shrink more than the quartz if they are cooled, and thus that if they are the same length, the temperature at which this occurs must be larger than the initial temperature. Next, we note that the steel rod melts first, and since the expansion is linear, it suffices to consider the lengths of the two rods at the melting point of steel. When we do so, we find that the steel rod is still shorter, which means that it must have been shorter at all lower temperatures as well. (In fact, the steel rod at its melting point is shorter than the length of the quartz rod at 0K if the formula still applied there, which it presumably does *not*.) So there is no temperature below the melting point of steel where the two rods could have been the same temperature, and furthermore, since the steel rod is shorter before it melts, the portion of it that remains solid could never be the same length as the quartz rod while it is melting (more on this below). Therefore **the solid rods are never the same length**.

The second solution (and the one most commonly given on the exam) is to solve explicitly for the temperature where the lengths would be equal, assuming the rods stayed in the solid phase. To do this, one sets

$$l_{0,q}(1 + \alpha_q \Delta T) = l_{0,s}(1 + \alpha_s \Delta T)$$

and then solves for ΔT, getting:

$$\Delta T = \frac{l_{0,q} - l_{0,s}}{l_{0,s}\alpha_s - l_{0,q}\alpha_q}$$

Plugging in the numbers gives about 1760.6°C. Recalling that $\Delta T = T_f - T_0$, we get a final temperature of $T_f = 1960.6$°C. This is above both melting points, so we believe the answer is **no**, but there is one more thing that I think needs to be said, which most people omitted. Namely the steel rod melts first and is shorter at its melting point (no calculation needed here because at any T between T_0 and T_f, the steel rod must be shorter, and we've already calculated that T_f is above the melting point of steel). But this means that as the steel melts, the portion of it that is in the solid phase will always be at least as short as the whole rod just before melting, and therefore shorter than the quartz rod. This may not seem important, but if I replaced quartz by an imaginary material with the same α and same initial length, but a lower melting point (below that of steel), then I would find that the solid lengths could be the same at some point while that imaginary material is melting.

Finally, the second solution can be modified to be substantially easier, calculation-wise. We observe from part a that the difference in lengths was reduced by 0.91 cm due to a temperature increase of 800 degrees. This means that a temperature increase of 1600 degrees would reduce the difference in length by less than 2 cm, not enough to close the gap between the initial lengths. But even $\Delta T = 1600°\text{C}$ puts the final temperature above both melting poins. From there, proceed as in solution 2.

1.4 Part b rubric

This question was worth the remaining 12 points. For solution 1, leaving out parts of the argument could result in small deductions, but people with the core argument that steel remains significantly shorter all the way to its melting point received full points. (I was very slightly more lenient with explanations for this answer, to reward the fact that it relies more heavily on physical understanding rather than mathematical manipulation.)

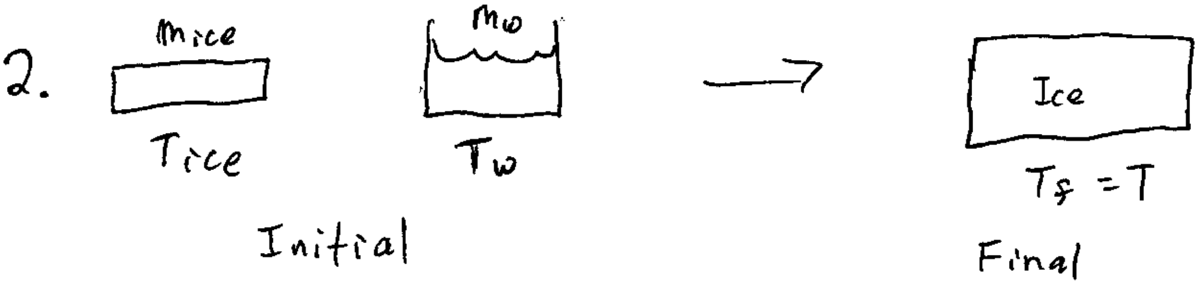
For the second answer, having the right formula at the beginning was worth 8 of the 12 points, although if it was then misapplied, some of these points were deducted. For instance: if in manipulation the formula was accidentally changed to a form where the units do not match up (such as $l_0 + \Delta T$), that was a 2 point deduction since units are important, and if the final ΔT was then used to calculate both $T_0 + \Delta T$ and $T_0 - \Delta T$, that was also a 2 point deduction (since ΔT is defined as $T_f - T_0$). Problems with the initial formula were larger deductions, typically 4 pts off.

The final number for T_f was then worth 1 point (this includes all mistakes with numbers, except for plugging in the wrong initial temperatures, which shows a more fundamental lack of understanding and was worth -2 points), having the correct conclusion based on the calculated temperature was an additional 1 point, and the explanation was 2 points, including 1 for the discussion of the lengths while the first one is melting. (Most people did not get this last point, and I would probably not have included it had I seen more exams before making my rubric, but the same standard applied to everyone, so it should not affect the curve.) Of these last 4 points, any final answer that viewed a temperature below 0 Kelvin as valid automatically lost all 4, and a temperature between the initial temperature and 1000 degrees automatically lost 2 (since you should know from part a that such an answer is not correct).

1.5 Other notes

Finally, I want to point out the most common mistakes to watch out for. First, 100 cm is 1 m, not 0.1 m. This was a surprisingly common mistake. Second, there were a lot of ridiculous numbers, like 80000 K, 50000000 K, and so forth. These are really ridiculous numbers, although I only took off 1 point for them since there is not any obvious physical reason why they are not possible answers. Still, you should recognize that such an answer must be incorrect. Next, it's much easier to get the right numerical answer to a problem if you plug in numbers when your equation looks like $\Delta T =$ something, rather than when it still has the variable you're solving for on both sides of the equation. Finally, please have a clear final answer. Some people arrived at the correct conclusion for part b, but the word "no" never appeared in the answer.

2.



There are four "Q's"

1. Ice goes up in temperature
2. Water goes down in Temp
3. water freezes
4. newly frozen water goes down in temp.

2 pts for identifying each → Total = $\boxed{8 \text{ pts}}$

$$\sum Q = 0 \quad \text{or} \quad Q_{in,ice} = Q_{out,w}$$ $\boxed{2 \text{ pts}}$

$$\overset{①}{m_{ice} \cdot C_{ice}(T - T_{ice})} + \overset{②}{m_w C_w (T_{freezing} - T_w)} + \overset{③}{(-1)\, m_w \cdot L_{ice}} + \overset{④}{m_w \cdot C_{ice} \cdot (T - T_{freezing})} = 0$$

$\boxed{1 \text{ pt}}$ correct signs

1 pt for each term being correct → Total = $\boxed{4 \text{ pts}}$

$T_{freezing} = 0°C = 0$ since we are in °C $\boxed{1 \text{ pt}}$

$$m_w = \frac{m_{ice} C_{ice} (T - T_{ice})}{C_w \cdot T_w + L_{ice} - C_{ice} \cdot T}$$ $\boxed{4 \text{ pts}}$ correct answer

Comments: Common scores: ≤ 15 WRONG SIGNS + ANSWER INCORRECT
≤ 13 Not identifying (4) (ANS INCORRECT + Not identifying 4 + correct writing of (4))

3.

masses are the same
$\Rightarrow$ call this M
identical containers
$\Rightarrow \mathcal{V}$ is the same

same temperatures T.

this is velocity, so the energy is always $\frac{3}{2}$. $\Rightarrow$ 1 pt
~~since distance [illegible]~~

a) $$\frac{v_{rms}^2 m}{2} = \frac{3}{2} kT \qquad \Rightarrow 3\text{pts}$$

solve: $$v_{rms} = \sqrt{\frac{3kT}{m}} \quad m_A = \frac{M}{N_A} \quad m_B = \frac{M}{N_B} \Rightarrow 2\text{ pts}$$

$$\frac{v_{rms}^A}{v_{rms}^B} = \sqrt{\frac{3kTN_A}{M}}\sqrt{\frac{M}{3kTN_B}} = \sqrt{\frac{N_A}{N_B}} \Rightarrow 2\text{pt}$$

b) Assume ideal gas:

$$P_A \mathcal{V}_A = N_A k T_A \quad P_B \mathcal{V}_B = N_B k T_B \Rightarrow \text{~~[illegible]~~ } 2\text{ pts}$$

$\mathcal{V}$ is the same $\Rightarrow$ ~~[illegible]~~ 1 pt

$$\frac{P_A}{P_B} = \frac{N_A k T_A}{\mathcal{V}_A} \frac{\mathcal{V}_B}{N_B k T_B} = \frac{N_A}{N_B} \quad \text{since } T_A = T_B \Rightarrow 2\text{ pts}$$

c) from a:
$$v_{rms}^A = \sqrt{\frac{3kT_A N_A}{M}} \quad v_{rms}^B = \sqrt{\frac{3kT_B N_B}{M}} \Rightarrow 3\text{ pts}$$

these are equal if $T_A N_A = T_B N_B \Rightarrow 2$ pt

$$T_A = \frac{T_B N_B}{N_A} \text{ and } T_B = T, \text{ so } T_A = \frac{TN_B}{N_A}$$

$$\frac{T_A - T}{T} = N_B/N_A - 1 \Rightarrow 2\text{ pts}$$

Total: 20 pts

Problem # 4

a)

$$\frac{dQ}{dT} = \left(\frac{k_1 2L}{2D} + \frac{k_2 3L}{2D}\right)(T - T_1) = 0.99\frac{L(T-20)}{D}$$

b)

$$\frac{dQ}{dT} = \left(\frac{k_3 5L}{D}\right)(T_2 - T) = 0.24\frac{L(-20-T)}{D}$$

c)

Set the previous answers equal as heat flow rate in equals the rate out and solve for T:

$$\left(\frac{k_1 2L}{2D} + \frac{k_2 3L}{2D}\right)(T - T_1) = \left(\frac{k_3 5L}{D}\right)(T_2 - T)$$

$$\left(\frac{2k_1 + 3k_2}{2} + 5k_3\right)T = 5k_3 T_2 + \frac{2k_1 + 3k_2}{2}T_1$$

$$T = \frac{10k_3 T_2 + (2k_1 + 3k_2)T_1}{2k_1 + 3k_2 + 10k_3} = 12.195°C$$

- Parts a) and b) have a total of 6 pts
  - (−1) if confusion between $P^{-\gamma}$ and $P^{1/\gamma}$
  - (−2) if final expression is given in terms of T ($TV^{\gamma-1} = cst$)

  Max of (2 pts) if $PV^{\gamma} = cst$ is stated and no answer.

- Part c) is worth 14 pts
  - (5) pts for Q_H computation (−3 pts if ΔE_{cd} or W_{cd} is wrong)
    (−1 pt if wrong sign)
    (−1 pt if algebra mistake with γ or d)
  - (1) pt for $e = \frac{W}{Q_H} = \frac{W_{tot}}{Q_{cd}}$ statement (or $e = 1 - \frac{Q_c}{Q_H} = 1 - \frac{(-Q_{ab})}{Q_{cd}}$)
  - (5) pts for Q_c computation <u>OR</u> the more painful W_{tot} computation
    (same breakdown as for Q_H) (3 pts for W_{tot} computation and plugging in e formula, and 2 pts for cancellations)
  - (3) pts for eliminating V_a, V_b, V_c, V_d from the expression for "e"

<u>General mistakes</u>:
- Forgetting terms in W_{tot} or in Q_H : −5 pts
- Using the variable "T" in Q_H and not switching back to P, V variables in the end: −2 pts
- Confusing Q_c and Q_H in the cycle: −4 pts
- Computation mistakes along the road: −1/−2 pts total
- Plugging in d=3 instead of keeping the variable "d": −1 pt
- CARNOT cycle efficiency instead: <u>1 pt Maximum</u> for part c)
- Using C_P in Q_H computation but no simplification: −2 pts
- Using C_V instead of C_P in Q_H: −3 pts

5. a) $b \to c$ is adiabatic, so $P_1 V_b^{\gamma} = P_2 V_c^{\gamma}$ $\left(\gamma = \frac{d+2}{d}\right)$

So $\boxed{\frac{V_b}{V_c} = \left(\frac{P_2}{P_1}\right)^{1/\gamma}}$

(6) pts

b) Same as a): $\boxed{\frac{V_a}{V_d} = \left(\frac{P_2}{P_1}\right)^{1/\gamma}}$

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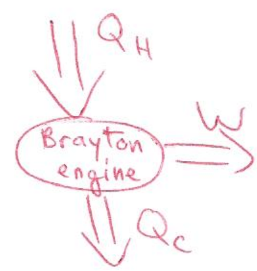
c) This is a heat engine, so $e = \frac{W_{\text{by gas}}}{Q_H}$



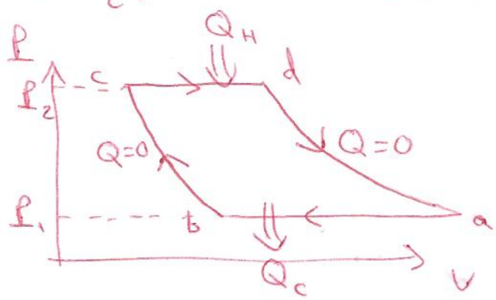
The "smart" way to compute e is to use:

$W = Q_H - Q_c$; then $e = 1 - \frac{Q_c}{Q_H}$ $\left(\begin{array}{l} Q_c > 0 \\ Q_H > 0 \end{array}\right)$

Use the first law to compute Q along each path:

- $Q_{bc} = Q_{da} = 0$ (adiabatic)
- $Q_{cd} = \Delta E_{cd} + W_{cd} = \frac{d}{2} P_2 \Delta V_{cd} + P_2 \int_{V_c}^{V_d} dV = P_2 (V_d - V_c)\left(1 + \frac{d}{2}\right) > 0$ since $V_d > V_c$
- $Q_{ab} = \Delta E_{ab} + W_{ab} = \frac{d}{2} P_1 \Delta V_{ab} + P_1 \int_{V_a}^{V_b} dV = P_1 (V_b - V_a)\left(1 + \frac{d}{2}\right) < 0$ since $V_b < V_a$

$\Rightarrow Q_H = Q_{cd} > 0$ and $Q_c = -Q_{ab} > 0$



So $e = 1 - \frac{Q_c}{Q_H} = 1 - \frac{P_1 (V_a - V_b)\cancel{(1 + d/2)}}{P_2 (V_d - V_c)\cancel{(1 + d/2)}}$

Use a) and b), $V_a - V_b = \left(\frac{P_2}{P_1}\right)^{1/\gamma} V_d - \left(\frac{P_2}{P_1}\right)^{1/\gamma} V_c$

So $e = 1 - \frac{P_1}{P_2}\left(\frac{P_2}{P_1}\right)^{1/\gamma}\left(\frac{\cancel{V_d - V_c}}{\cancel{V_d - V_c}}\right) \Rightarrow \boxed{e = 1 - \left(\frac{P_1}{P_2}\right)^{\frac{\gamma - 1}{\gamma}}}$ (Note $e < 1$)

(14) pts

~~[illegible]~~ Note: We could have also computed W for all paths and then used $e = \frac{W}{Q_H}$, but this involves nasty integrals!

$W_{bc} = \frac{P_1 V_b^{\gamma}}{1-\gamma}\left(V_c^{1-\gamma} - V_b^{1-\gamma}\right)$ and $W_{da} = \frac{P_1 V_a^{\gamma}}{1-\gamma}\left(V_a^{1-\gamma} - V_d^{1-\gamma}\right)$ for adiabats.

The other two isobaric paths are easy: $W_{cd} = P_2 (V_d - V_c)$ and $W_{ab} = P_1 (V_b - V_a)$

The sum $W_{tot} = \left(P_2 (V_d - V_c) - P_1 (V_a - V_b)\right)\frac{\gamma}{\gamma - 1}$

(Use $\frac{\gamma}{\gamma - 1} = \frac{d}{2} + 1$ to simplify "e")