UCB Math 1B, Fall 2011: Midterm 1

Prof. Persson, October 4, 2011

Name: Solutions

SID: ______

Section: Circle your discussion section below:

Sec	Time	Room	GSI
01	MWF 8am - 9am	3 Evans	H. Lee
02	MWF 4pm - 5pm	45 Evans	A. Lieb
03	MWF 9am - 10am	6 Evans	H. Lee
04	MWF 10am - 11am	3111 Etcheverry	Z. Rosen
05	MWF 11am - 12pm	3111 Etcheverry	J. Chen
06	MWF 12pm - 1pm	3111 Etcheverry	Z. Rosen
07	MWF 1pm - 2pm	310 Hearst	W. H. Cook
08	MWF 2pm - 3pm	285 Cory	W. H. Cook
09	MWF 3pm - 4pm	35 Evans	J. Chen
10	MWF 4pm - 5pm	2 Evans	C. Y. Cho
12	MWF 5pm - 6pm	45 Evans	C. Y. Cho
13	MWF 11am - 12pm	230D Stephens	A. Adiredja

Other/none, explain: ______

Grading

1	/9
2	/9
3	/5
4	/5
5	/5
	/33

Instructions:

- One double-sided sheet of notes, no books, no calculators.
- Exam time 80 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. Evaluate the integral or show that it is divergent.

a) (3 points) $\int \frac{\sin 2\theta}{\sin^2\theta + 1}\, d\theta$

method 1

$= \int \frac{\sin 2\theta}{\frac{1-\cos 2\theta}{2} + 1}\, d\theta$

$= \int \frac{2\sin 2\theta}{3 - \cos 2\theta}\, d\theta$ $\left[\begin{array}{l} u = 3 - \cos 2\theta \\ du = 2\sin 2\theta\, d\theta \end{array}\right]$

$= \int \frac{du}{u}$

$= \ln|3 - \cos 2\theta| + C$

method 2

$= \int \frac{2\sin\theta\cos\theta\, d\theta}{\sin^2\theta + 1}$ $\left[\begin{array}{l} u = \sin\theta \\ du = \cos\theta\, d\theta \end{array}\right]$

$= \int \frac{2u\, du}{u^2+1}$ $\left[\begin{array}{l} w = u^2+1 \\ dw = 2u\, du \end{array}\right]$

$= \int \frac{dw}{w}$

$= \ln|w| + C$

$= \ln(\sin^2\theta + 1) + C$

b) (3 points) $\int_0^\infty \frac{\arctan x}{1+x^2}\, dx$

$\left[\begin{array}{l} u = \arctan x \\ du = \frac{1}{1+x^2}\, dx \end{array}\right]$

$= \lim_{t\to\infty} \int_0^t \frac{\arctan x}{1+x^2}\, dx = \lim_{t\to\infty} \int_0^{\arctan t} u\, du = \lim_{t\to\infty} \frac{u^2}{2}\Big|_0^{\arctan t}$

$= \lim_{t\to\infty}\left(\frac{(\arctan t)^2}{2} - \cancel{\frac{(\arctan(0))^2}{2}}^{\,0}\right) = \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi^2}{8}$

c) (3 points) $\int \sec^6 x\, dx$

$= \int \sec^4 x \sec^2 x\, dx$

$= \int (1 + \tan^2 x)^2 \sec^2 x\, dx$ $\left[\begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array}\right]$

$= \int (1 + 2\tan^2 x + \tan^4 x) \sec^2 x\, dx$

$= \int (1 + 2u^2 + u^4)\, du$

$= u + \frac{2u^3}{3} + \frac{u^5}{5} + C$

$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

2. For the series below, find the sum or show that it is divergent.

a) (3 points) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

COMPARISON TEST

Let $a_n = \dfrac{\ln n}{n}$, $b_n = \dfrac{1}{n}$. Both a_n and b_n are positive, and $a_n > b_n$ for all $n > e$, i.e. $n \geq 3$. So, because $\sum_{n=3}^{\infty} b_n$ = the harmonic series diverges, $\sum_{n=3}^{\infty} a_n$ diverges.

Therefore, $\sum_{n=2}^{\infty} a_n$ diverges as well.

b) (3 points) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n + 1}{3^n}$

By the series laws, if $\sum a_n$ and $\sum b_n$ are convergent, then $\sum (a_n + b_n) = \sum a_n + \sum b_n$.

Here, $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{e}{3}\left(\frac{e}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$

These are geometric series, so the formula gives

$$\sum_{n=1}^{\infty} \frac{e^n+1}{3^n} = \frac{e/3}{1-e/3} + \frac{1/3}{1-1/3} = \frac{e}{3-e} + \frac{1}{2} = \frac{e+3}{6-2e}$$

c) (3 points) $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - 1\right)$

LIMIT COMPARISON TEST

Let $a_n = e^{1/n} - 1$ and $b_n = \dfrac{1}{n}$. $e^{1/n} > e^0 = 1$, so a_n and b_n are both positive.

Calculate: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{1/n}-1}{1/n}$. Since numerator and denominator approach 0, we can apply L'Hopital's Rule to the corresponding continuous functions.

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{\left(-\frac{1}{x^2}\right) e^{1/x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty} e^{1/x} = e^{\lim_{x\to\infty} 1/x} = e^0 = 1 > 0.$$

So, by LCT, since $\sum b_n$ diverges, $\sum a_n$ diverges as well.

3. **a)** (3 points) Use Simpson's Rule to approximate the integral $\int_0^1 \frac{1}{1+x^2}\,dx$ using $n = 2$ subintervals.

$$S_2 = \frac{\Delta x}{3}\left[f(0) + 4f(\tfrac{1}{2}) + f(1)\right]$$ our $\Delta x = 1/2$ since $n=2$

$$= \frac{1}{6}\left[1 + 4\left(\frac{4}{5}\right) + \frac{1}{2}\right].$$

$$= \frac{1}{6}\left[\frac{5+10+32}{10}\right] = \frac{47}{60}$$

b) (2 points) Use your result in **(a)** to approximate π.

$$\int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \pi/4.$$

We know $S_2 \approx \int_0^1 \frac{dx}{1+x^2}$

$$\Rightarrow \frac{47}{60} \approx \pi/4 \Rightarrow \pi \approx \frac{47}{15}$$

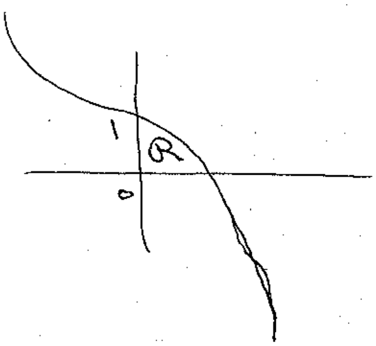
4. (5 points) Find the centroid of the region bounded by the curves $x = 1 - y^3$, $x = 0$, and $y = 0$.

Recall that $\bar{y} = \frac{1}{A}\int yx\,dy$ and $\bar{x} = \frac{1}{A}\int \frac{1}{2}x^2\,dy$

where $A = \int x\,dy$ with appropriate limits defined by the region.

~~So the~~

The region is as follows.



So $A = \int_0^1 x\,dy = \int_0^1 (1-y^3)\,dy = y - \frac{y^4}{4}\Big]_0^1 = \frac{3}{4}$

Also, $\bar{y} = \frac{1}{A}\int_0^1 yx\,dy$

$= \frac{1}{A}\int_0^1 y(1-y^3)\,dy$

$= \frac{1}{A}\left(\frac{1}{2}y^2 - \frac{1}{5}y^5\right)\Big]_0^1 = \frac{1}{A}\frac{3}{10} = \frac{2}{5}$

and $\bar{x} = \frac{1}{A}\int_0^1 \frac{1}{2}x^2\,dy$

$= \frac{1}{A}\int_0^1 \frac{1}{2}(1-y^3)^2\,dy = \frac{1}{2A}\left(y - \frac{2}{4}y^4 + \frac{1}{7}y^7\right)\Big]_0^1$

$= \frac{1}{2A}\cdot\frac{9}{14} = \frac{3}{7}$

So the centroid is $(\bar{x}, \bar{y}) = (3/7, 2/5)$

(You may compute in terms of function of x)

5. (5 points) Evaluate the integral: $\int \ln\left[(x^2-2x+3)^2\right]\,dx$

$$= 2\int \ln(x^2-2x+3)\,dx = [\text{Int. by parts}]$$

$$= 2x\cdot\ln(x^2-2x+3) - 2\underbrace{\int \frac{x(2x-2)}{x^2-2x+3}\,dx}_{I}$$

$$\begin{array}{r|l} & \;2 \\ \hline x^2-2x+3 & \;2x^2-2x \\ & -(2x^2-4x+6) \\ \hline & \;2x-6 \end{array}$$

$$I = \int\left[2 + \frac{2x-6}{x^2-2x+3}\right]dx = 2x + \int\frac{2x-2}{x^2-2x+3}\,dx - \int\frac{4}{(x-1)^2+2}\,dx$$

$$= 2x + \ln(x^2-2x+3) - \frac{4}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{2}} + C$$

$$\Rightarrow \int \ln\left[(x^2-2x+3)^2\right]dx = 2(x-1)\cdot\ln(x^2-2x+3)$$

$$-4x + 4\sqrt{2}\tan^{-1}\frac{x-1}{\sqrt{2}} + C$$