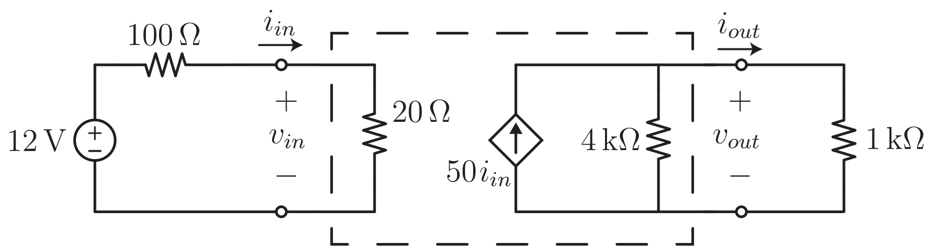
Midterm Exam Solutions

1. **Answer ALL the questions!**

   (a) *(5 points)* In the following circuit, we pass an input signal through a current amplifier. Evidently, it is nonideal with finite input and output impedances. The output is attached to a load $R_L = 1\,\text{k}\Omega$.



i. Calculate the power P_{in} at the input of the amplifier.

$$v_{in} = \frac{20}{100+20} \times 12\,\text{V} = 2\,\text{V}$$

$$P_{in} = \frac{v^2}{R} = \frac{2^2}{20} = \boxed{0.2\,\text{W}}$$

ii. Calculate the power P_{out} at the output of the amplifier.

$$i_{in} = \frac{12\,\text{V}}{100+20} = 0.1\,\text{A}$$

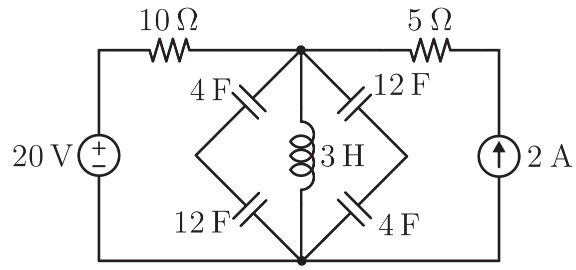
$$i_{out} = \frac{4}{1+4} \times 50i_{in} = \frac{4}{5} \times 5 = 4\,\text{A}$$

$$P_{out} = i^2R = 4^2 \times 1\,\text{k}\Omega = \boxed{16\,\text{kW}}$$

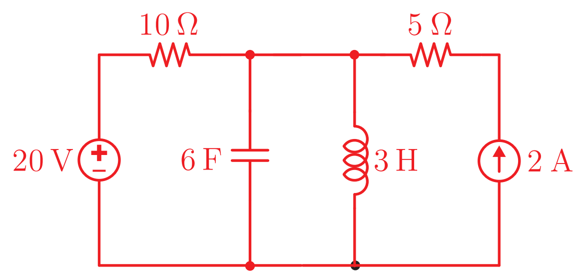
iii. What is the power gain G of the amplifier?

$$G = \frac{P_{out}}{P_{in}} = \frac{16\,\text{kW}}{0.2\,\text{W}} = \boxed{80000}$$

(b) *(10 points)* Consider the following circuit.



i. Simplify the circuit by applying all possible series and parallel combinations to any resistors, capacitors, inductors.



ii. Assuming DC steady state, calculate the powers of the sources and state whether each is supplying or absorbing energy.

At DC steady state, capacitor is treated as open circuit and inductor is treated as short circuit.

For the voltage source: $P = VI = 20 \times -\frac{20}{10} = \boxed{-40\,\text{W},\ \textit{supplying}}$

For the current source: $P = VI = -5 \times 2 \times 2 = \boxed{-20\,\text{W},\ \textit{supplying}}$

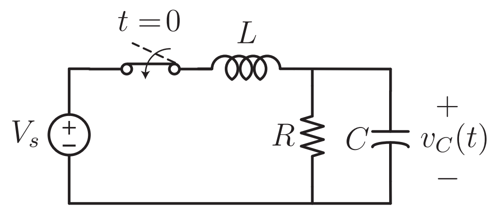
iii. Find the energy stored in the inductor assuming DC steady state conditions.

At DC steady state, inductor is fully charged. The total current going through inductor is

$$i = \frac{20}{10} + 2 = 4\,\text{A}$$

$$E = \frac{1}{2}LI^2 = \frac{1}{2} \times 3 \times 4^2 = \boxed{24\,\text{J}}$$

(c) *(10 points)* Consider the following RLC circuit. The capacitor is initially uncharged. Express all results below in terms of the given circuit parameters.



i. Derive the ODE governing $v_C(t)$ for $t > 0$ (V_s is a DC source).

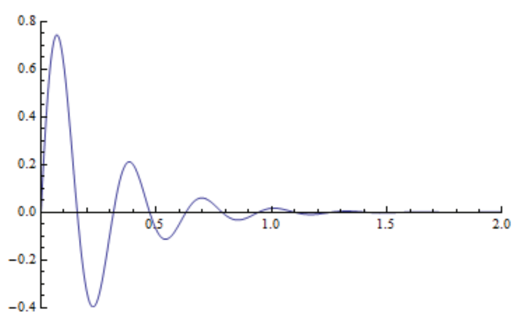
KCL at v_c:

$$C\frac{d(v_C(t)-0)}{dt} + \frac{v_C(t)-0}{R} + \frac{1}{L}\int (v_C(t)-V_s)dt = 0$$

$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} + \frac{1}{LC}\int (v_C(t)-V_s)dt = 0$$

$$\boxed{\frac{d^2v_C(t)}{dt^2} + \frac{1}{RC}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC}}$$

ii. Assuming *underdamped* behavior, qualitatively describe the behavior of $v_C(t)$ over time. You may sketch plots to supplement your explanation.



Note the scale in the figure above is arbitrary.

Underdamped behavior leads to damped oscillations, usually with an initial overshoot followed by ringing before settling down to a steady-state value.

iii. Find an expression for the energy stored in the capacitor after a very long time.

At DC steady state, capacitor is treated as open circuit and inductor is treated as short circuit. So,

$$v_C(\infty) = V_s$$

$$E_{cap} = \frac{1}{2}CV_C(\infty)^2 = \boxed{\frac{1}{2}CV_s^2}$$

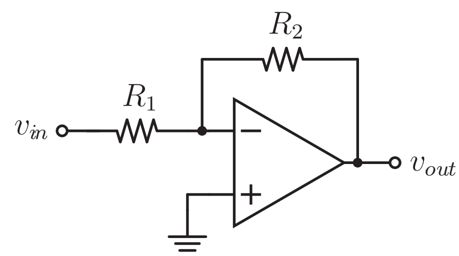
2. **OVER 9000!!!**

Today you will be helping Jerry design an amplifier circuit to meet some specifications that Tony has given him (be sure to use only KCL or Jerry will not be very happy).

(a) *(1 point)* Tony wants as much gain as possible. Since an ideal op amp has infinite open-loop gain, Jerry does not see why negative feedback would be useful, seeing as it drastically reduces the gain. Name one benefit of negative feedback, other than the ability to apply the summing-point constraint.

Robustness, Stability, Dynamic input range, etc.

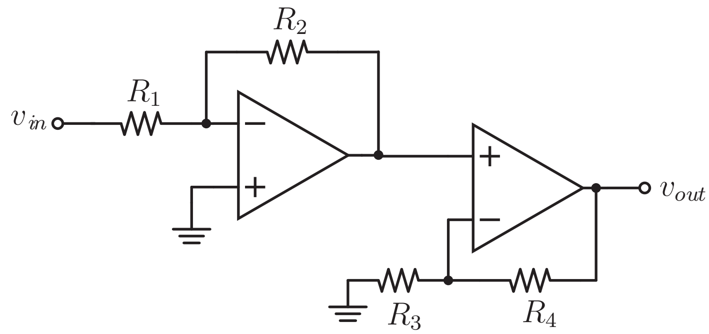
(b) *(2 points)* Jerry believes you and agrees to use negative feedback configurations. Tony wants a gain of over -9000, so Jerry confidently designs an inverting amplifier with a gain of -10000, shown below. Help Jerry choose resistor values such that he achieves that gain.



This is an inverting op-amp configuration. The gain is simply $G = -\frac{R_2}{R_1}$.

Choosing $R_1 = 10\,\Omega, R_2 = 100\,\mathrm{k}\Omega; G = -\frac{R_2}{R_1} = -\frac{100\,\mathrm{k}\Omega}{10\,\Omega} = -10000.$

(c) *(4 points)* Now Tony comes over and says that the lab only has resistor values between $100\,\Omega$ and $10\,\text{k}\Omega$ due to budget cuts. "No problem!" says Jerry. He decides to add a noninverting stage after the first amplifier. Choose new values for all resistors so that we still have a gain of over -9000.



This is cascade of inverting op-amp and non-inverting op-amp. The gain is

$$G = G_{inv} \times G_{non-inv} = -\frac{R_2}{R_1} \times (1 + \frac{R_4}{R_3})$$

Choosing $R_1 = R_3 = 100\,\Omega$, and $R_2 = R_4 = 10\,\text{k}\Omega$, the gain of overall circuit is

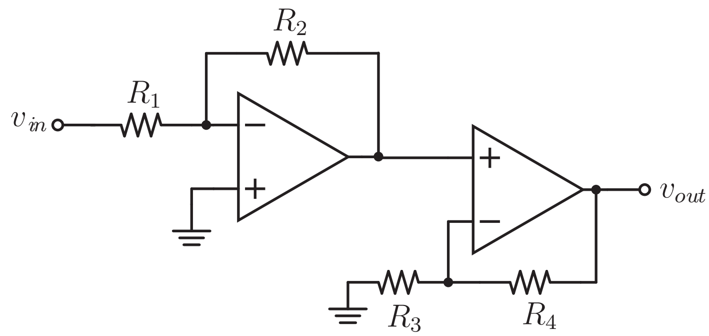
$$G = -\frac{R_2}{R_1}(1 + \frac{R_4}{R_3}) = -\frac{10\,\text{k}\Omega}{100\,\Omega}(1 + \frac{10\,\text{k}\Omega}{100\,\Omega}) = -10100$$

Note: Any value of resistors for (a) and (b) is acceptable as long as gain satisfies the requirement.

(d) *(2 points)* Jerry builds his circuit and finds that it doesn't work. But of course, he has forgotten to power the amplifier! He attaches supply rails of $\pm 15\,\text{V}$ to both op amps. Using the resistance values that you chose above, what is the voltage input range for which amplifier operation remains linear?

$$-\frac{15\,\text{V}}{G} \le v_{in} \le \frac{15\,\text{V}}{G}$$

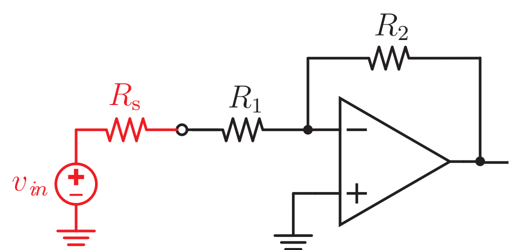
$$\boxed{-\frac{15}{10100}\,\text{V} \le v_{in} \le \frac{15}{10100}\,\text{V}}$$



(e) *(3 points)* Just as Jerry finishes his new design, Tony comes over and says, "Oh btw, the source resistances for the inputs will be on the order of $10\,\text{k}\Omega$." Find the input resistance of the circuit using your resistance values and explain why this is problematic.

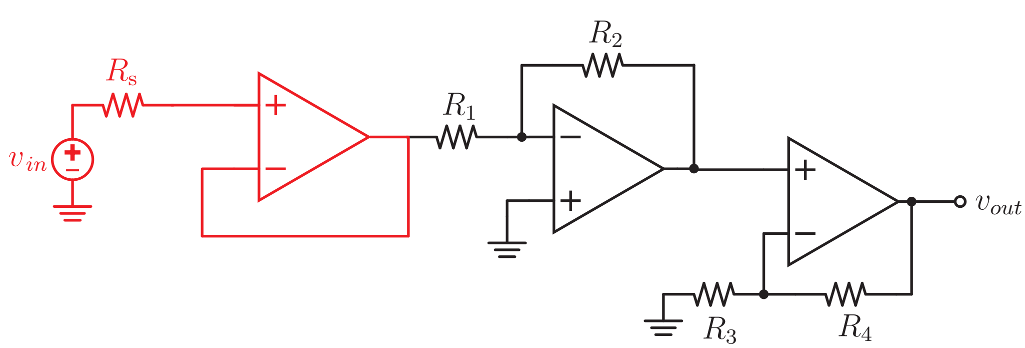
The input resistance of the amplifier can be found by applying test source at the input,

$$\frac{v_{test} - 0}{R_1} = i_{test}, R_{in} = \frac{v_{test}}{i_{test}} = \boxed{R_1 = 100\,\Omega}$$



We can model the source as shown above. Solving the circuit with SPC and KCL, the gain of the first stage (inverting op-amp configuration) becomes $G_{inv} = -\dfrac{R_2}{R_1 + R_s} = -\dfrac{10\,\text{k}\Omega}{100\,\Omega + 10\,\text{k}\Omega} \approx -1$. Therefore, the total gain drops significantly.

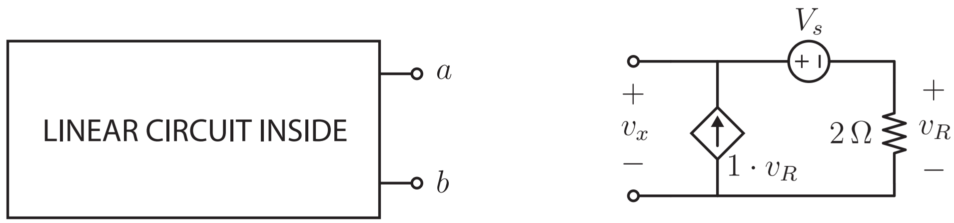
(f) *(3 points)* Tired of the constant spec changes, Jerry ragequits and leaves the problem for Dennis to fix. Help Dennis rectify the new problem by adding another stage to the amplifier circuit to make the input resistance as high as possible without changing the gain.



To solve the problem of impedance matching, we can insert a voltage follower as shown above (infinite input resistance with a gain of 1, ideally) at the beginning of the circuit. Now the gain is not affected by the source resistance and stays at -10100. The tradeoff of adding another stage, of course, is that the circuit now consumes more power and takes up more space.

3. **Probing Thévenin with a Device (twss)**

William has stumbled across a mysterious electronic box labeled "LINEAR CIRCUIT INSIDE." Hearing a ticking sound coming from inside, he has decided that it would be unsafe to open it up. However, he's still very curious about what the circuit looks like inside, so he probes the two terminals sticking out of the box (for some reason this seemed perfectly okay, even though opening it was not).



Instead of having a multimeter like normal people, John has the above whack measuring device (shown on the right). The device can be attached across two terminals of any circuit. The value of V_s can be twiddled with, and the device will then report the voltage v_x.

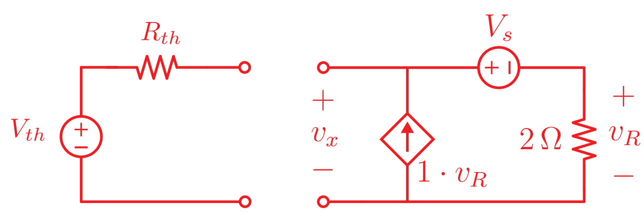
(a) *(12 points)* William decides to use John's device to probe the box. He makes two measurements and observes the following:

$$V_s = 5\,\mathrm{V} \rightarrow v_x = -3\,\mathrm{V}$$
$$V_s = 6\,\mathrm{V} \rightarrow v_x = 0\,\mathrm{V}$$

Using this information, help William find the Thévenin and Norton equivalent circuits of the box.

Hint: Pick one equivalent circuit to focus on first. You can get the second one easily afterward.

Because we know the circuit inside is linear, we are able to draw out an equivalent circuit and attach that to our "whack" device. This is shown below for a Thévenin device, but you can also do this for a Norton device.



Now we are able to use nodal analysis to find V_{th} and R_{th}. Using a supernode, we get the following two equations:

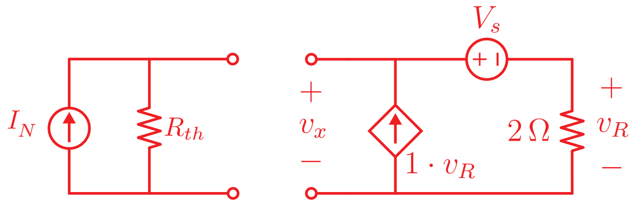
$$\frac{v_x - V_{th}}{R_{th}} + \frac{V_R}{2} - 1 \cdot v_R = 0$$
$$v_x - V_R = V_s$$

Using your two conditions and substituting in the supernode equation, you can get the following equations:

$$\frac{-3\,\mathrm{V} - V_{th}}{R_{th}} + \frac{-8\,\mathrm{V}}{2\,\Omega} - -8\,\mathrm{A} = 0$$
$$\frac{-V_{th}}{R_{th}} + \frac{-6\,\mathrm{V}}{2\,\Omega} - -6\,\mathrm{A} = 0$$

Your second equation gives you $\boxed{I_N = \frac{V_{th}}{R_{Th}} = 3\,\mathrm{A}}$, and plugging $V_{th} = 3R_{Th}$ in, you are able to get $\boxed{R_{th} = 3\,\Omega}$. Because $V = IR$, $\boxed{V_{th} = 9\,\mathrm{V}}$.

More space for part (a)...



Here we will do the Norton equivalent circuit. The two equations from the supernode around V_s is as follows:

$$\frac{v_x}{R_{th}} + \frac{V_R}{2} - I_N - 1 \cdot v_R = 0$$

$$v_x - V_R = V_s$$

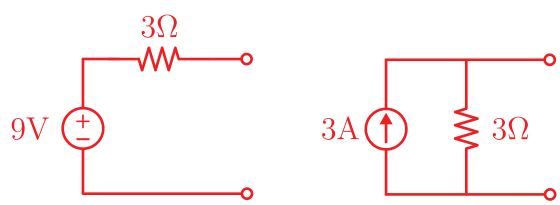
From your two conditions, you will get the following equations after substituting in the supernode equation:

$$I_N + \frac{-3\,\text{V}}{R_{th}} + \frac{-8\,\text{V}}{2\,\Omega} - -8\,\text{A} = 0$$

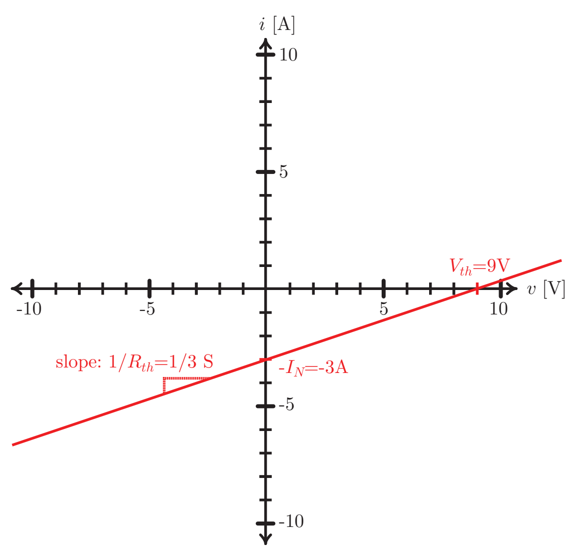
$$I_N + \frac{-6\,\text{V}}{2\,\Omega} - -6\,\text{A} = 0$$

As before, you should get, $\boxed{I_N = 3\,\text{A}}$, $\boxed{R_{th} = 3\,\Omega}$, and $\boxed{V_{th} = 9\,\text{V}}$

The equivalent circuits are shown in the figure below:

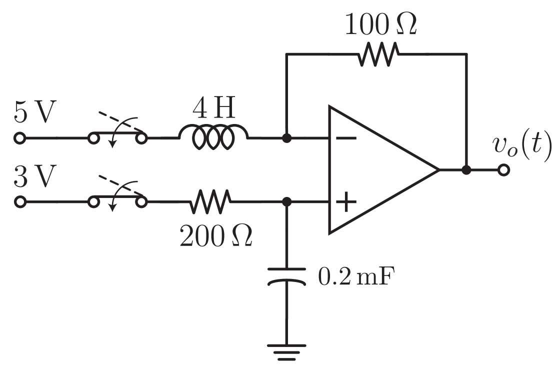


(b) *(3 points)* While William has only performed two measurements, he now essentially knows the black box's current output for any voltage input (or vice-versa). For the references shown below, plot the box's i-v characteristic. Label axes, intercepts, and slopes.



4. **What Is This I Don't Even**

In an attempt to troll the students, Dennis has constructed the following monstrosity. Both switches close at $t = 0$, and the capacitor is uncharged prior to the switch actions. **Assume that the op amp is ideal with no rail limitations and that negative feedback holds.**



(a) *(8 points)* Find $v_+(t)$ at the noninverting input for $t > 0$.

Because negative feedback is present, we can apply the summing point constraint. Hence, no current goes into the noninverting terminal of the op amp, and we simply have our canonical RC charging circuit. Writing KCL at $v_+(t)$ gives us

$$\frac{v_+(t) - 3}{200} + 0.2 \times 10^-3 \frac{dv_+(t)}{dt} = 0$$

$$v_+(t) + 0.04 \frac{dv_+(t)}{dt} = 3 \tag{1}$$

Hence, the time constant for the complementary solution is $\tau = 0.04$. Notice that our particular solution is $f(t) = 3$, so the particular solution is a constant. Our full solution assumes the form

$$v_+(t) = Ke^{-t/\tau} + A \tag{2}$$

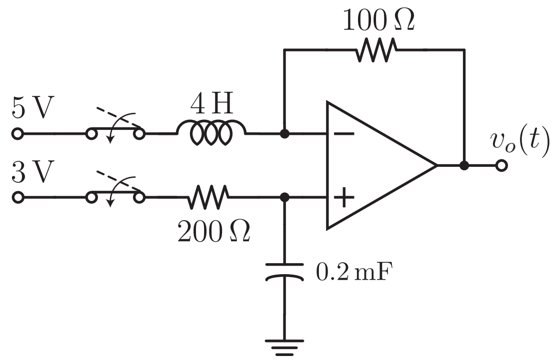
We apply the usual boundary conditions to solve for the constants. Before the switch closes, the capacitor is uncharged, so we infer that $v_+(0) = 0$. After a long time, the capacitor becomes an open circuit in DC steady state, so $v_+(\infty) = 3$ from the input source. Hence,

$$v_+(0) = 0 = K + A$$

$$v_+(\infty) = 3 = A$$

Solving for both constants gives us the solution

$$v_+(t) = 3(1 - e^{-25t})\,\text{V} \tag{3}$$



(b) *(3 points)* Write an equation for $v_o(t)$ in terms of $v_-(t)$, the voltage at the inverting input.

We apply KCL at the inverting input and rearrange to obtain an expression for $v_o(t)$. Be careful of the limits of the integral!

$$\frac{1}{4}\int_0^t v_-(t) - 5\,\mathrm{d}t + \frac{v_-(t) - v_o(t)}{100} = 0$$

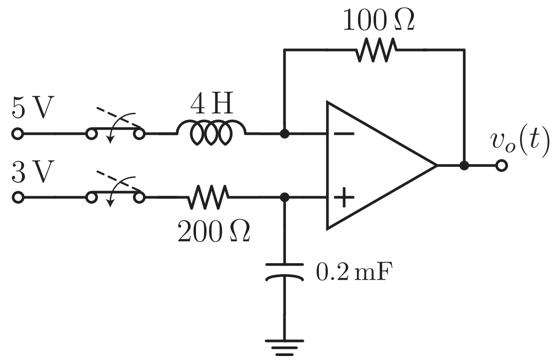
$$v_o(t) = 25\int_0^t v_-(t) - 5\,\mathrm{d}t + v_-(t) \tag{4}$$

(c) *(6 points)* Use your results from (a) and (b) to solve for $v_o(t)$.

The summing point constraint tells us that $v_+(t) = v_-(t)$, so we can plug our solution from Eqn 3 into the above expresssion 4 for $v_o(t)$.

$$\begin{aligned} v_o(t) &= 25\int_0^t 3(1 - e^{-25t}) - 5\,\mathrm{d}t + 3(1 - e^{-25t}) \\ &= 25\left(-2t + \frac{3e^{-25t}}{25}\right)\Bigg|_0^t + 3(1 - e^{-25t}) \\ &= -50t + 3e^{-25t} - 3 + 3 - 3e^{-25t} \end{aligned}$$

$$v_o(t) = -50t\,\mathrm{V} \tag{5}$$



(d) *(4 points)* Does $v_-(t)$ ever reach a steady-state value? If so, find the value; if not, explain why this circuit does not allow for this to happen.

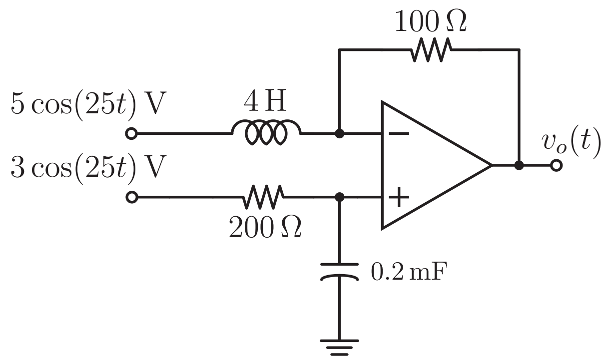
Yes, it does. By negative feedback and the summing point constraint, the voltage v_- must be equal to the voltage v_+. We know that $v_+(t)$ reaches a steady state value from Eqn 3, so $v_-(t)$ must also reach the same value at 3 V.

(e) *(4 points)* Does $v_o(t)$ ever reach a steady-state value? If so, find the value; if not, explain why this circuit does not allow for this to happen.

No, it does not. Notice from our expression for $v_o(t)$ from Eqn 5 that it is a linearly decreasing term as time goes to ∞. This can be understood from the constant voltage drop across the inductor, hence leading to a linearly ramping current across it and to the output node. Because current across the 100-Ω resistor is ramping as well, $v_o(t)$ never reaches a steady state value.

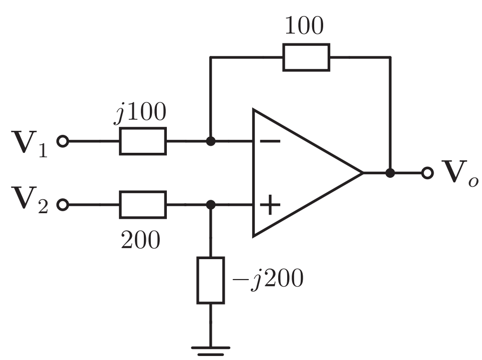
5. **Not Sure If Wrong Figure... Or If GSIs Just Got Lazy**

John takes Dennis's circuit and replaces the input voltages with sinusoidal sources, both with frequency $\omega = 25$ rad/sec. He also wants to find $v_o(t)$, though he doesn't care about the transient response.



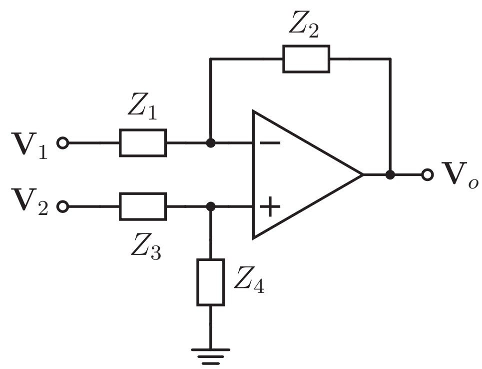
(a) *(5 points)* Help John thwart Dennis's ODE trolling plan by redrawing the circuit with all elements converted to the phasor domain.

Converting to the phasor domain, we have



where $\mathbf{V}_1 = 5\angle 0$ and $\mathbf{V}_2 = 3\angle 0$

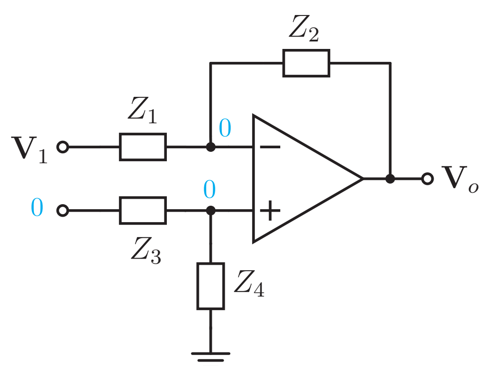
(b) *(7 points)* Dennis accidentally lets slip that this amplifier is actually a difference amplifier, though with impedances instead of resistances. John takes the opportunity to redraw the circuit as follows:



Show that the gain for a general difference amplifier is

$$\mathbf{V}_o = \left(\frac{Z_4}{Z_3 + Z_4}\right)\left(\frac{Z_1 + Z_2}{Z_1}\right)\mathbf{V}_2 - \left(\frac{Z_2}{Z_1}\right)\mathbf{V}_1$$

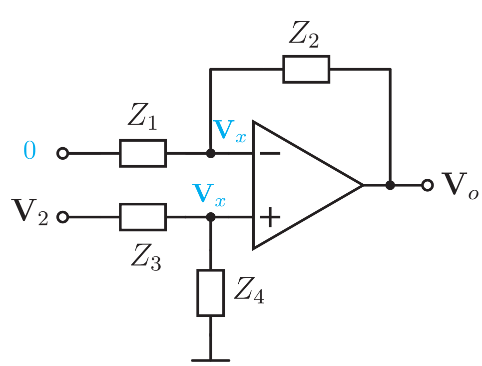
Since this is a summing and difference amplifier, we will use superposition as described in lecture. First we will turn off $\mathbf{V}_2$. Therefore, by the summing point constraint, the nodal voltage at both terminals become 0 V, as shown.



This is simply an inverting amplifier. Therefore, the contribution from $\mathbf{V}_1$ to $\mathbf{V}_0$ is determined to be:

$$\mathbf{V}_{02} = -\mathbf{V}_1 \frac{Z_2}{Z_1}$$

Next, we will turn off $\mathbf{V}_1$.



We recognize that this is a non-inverting amplifier and therefore the gain is determined to be:

$$\mathbf{V}_{01} = \mathbf{V}_x \left(1 + \frac{Z_2}{Z_1}\right)$$

where $\mathbf{V}_x$ is the nodal voltage at both the inverting and non inverting terminal by summing point constraint. Therefore, by voltage divider, it is:

$$\mathbf{V}_x = \mathbf{V}_2 \left(\frac{Z_4}{Z_4 + Z_3}\right)$$

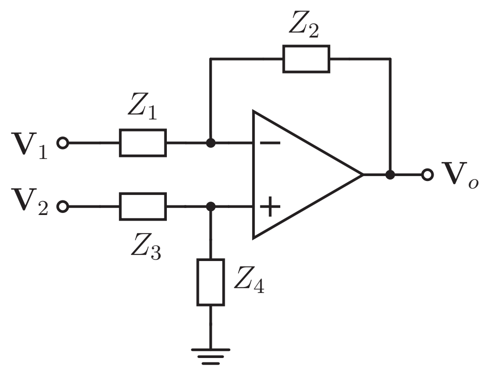
Therefore, the contribution made by $\mathbf{V}_2$ is:

$$\mathbf{V}_{02} = \mathbf{V}_2 \left(\frac{Z_4}{Z_4 + Z_3}\right) \left(1 + \frac{Z_2}{Z_1}\right)$$

We can also arrive at this expression had we done KCL at the inverting terminal.

As a result, the output voltage is the sum of these two sources' contributions by superposition. The final expression for output is:

$$\boxed{\mathbf{V}_o = \left(\frac{Z_4}{Z_3 + Z_4}\right) \left(\frac{Z_1 + Z_2}{Z_1}\right) \mathbf{V}_2 - \left(\frac{Z_2}{Z_1}\right) \mathbf{V}_1} \tag{6}$$



$$\mathbf{V}_o = \left(\frac{Z_4}{Z_3 + Z_4}\right)\left(\frac{Z_1 + Z_2}{Z_1}\right)\mathbf{V}_2 - \left(\frac{Z_2}{Z_1}\right)\mathbf{V}_1$$

(c) *(4 points)* Using the gain given above, derive an expression for $\mathbf{V}_o$, the phasor form of $v_o(t)$.

Plugging in the complex impedances from part (a), we get:

$$\begin{aligned}
\mathbf{V}_o &= \left(\frac{-200j}{200 - 200j}\right)\left(\frac{100j + 100}{100j}\right)\mathbf{V}_2 - \frac{100}{100j}\mathbf{V}_1 \\
&= \left(\frac{-j}{1-j}\right)\left(\frac{j+1}{j}\right)\mathbf{V}_2 - \frac{1}{j}\mathbf{V}_1 \\
&= -\left(\frac{1+j}{1-j}\right)\mathbf{V}_2 + j\mathbf{V}_1 \\
&= -\left(\frac{1+j}{1-j}\right) \times \left(\frac{1+j}{1+j}\right)\mathbf{V}_2 + j\mathbf{V}_1 \\
&= -\left(\frac{2j}{2}\right)\mathbf{V}_2 + j\mathbf{V}_1 \\
&= j(\mathbf{V}_1 - \mathbf{V}_2) \\
&= \left(1\angle\frac{\pi}{2}\right)(5\angle 0 - 3\angle 0) \\
&= 2\angle\frac{\pi}{2}
\end{aligned}$$

(d) *(4 points)* Perform the final conversion back to the time domain to obtain an expression for $v_o(t)$.

The phasor represents a scaled and shifted version of $\cos(\omega t)$.

$$\begin{aligned}
v_o(t) &= 2\cos(\omega t + \frac{\pi}{2}) \\
&= 2\sin(\omega t + \frac{\pi}{2} + \frac{\pi}{2}) \\
&= 2\sin(\omega t + \pi) \\
&= -2\sin(\omega t)
\end{aligned}$$