CS 70 Final 9 August 2012

NAME (1 pt): ______________________________

SID (1 pt): ______________________________

TA (1 pt): ______________________________

Name of Neighbor to your left (1 pt): ______________________________

Name of Neighbor to your right (1 pt): ______________________________

Instructions: This is a closed book, closed calculator, closed computer, closed network, open brain exam, but you are permitted a 3 page, double-sided set of notes, large enough to read without a magnifying glass.

You get one point each for filling in the 5 lines at the top of this page.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

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Total	

Question 1 (20 points) True or False.

For each of the following propositions, circle either T if it is always true, F if it is always false. You do not have to justify your answer.

We let $\mathbb{N} = \{0, 1, 2, ...\}$ denote the non-negative integers, $\mathbb{Q}$ denote the set of rational numbers. A and B denote events in the sample space S.

T F For all propositions P, $[P(0) \wedge P(1) \wedge (\forall k \in \mathbb{N} P(k) \implies P(k+2))] \implies \forall n \in \mathbb{N} P(n)$

T F Here P and Q denote propositions, each of which could be true or false: $[P \implies (Q \wedge \neg Q)] \implies \neg P$

T F Given a list of x values for a polynomial and the delta functions generated from Lagrange Interpolation, they can be used to pass through an arbitrary combination of y values.

T F There are a total of p^{d+1} polynomials of degree d in $GF(p)$

T F For a uniformly distributed random variable defined on $[a, b]$, it's sufficient to know the expectation and variance of the random variable to uniquely determine the intervals a and b.

T F There are $\binom{n+k}{k}$ different ways to throw k identical balls into n distinguishable bins. Each bin can have multiple balls.

T F Markov's inequality, Chebyshev and the Central Limit Theorem all are useful for upper bounding hard to compute probabilities.

T F The multiplicative inverse of a modulo M exists $\iff$ M is prime.

T F The set of degree n polynomials over $\mathbb{Q}$ is countable

T F The set of numbers of the form $(\sqrt{x} + \sqrt{y})^2$ where $x, y \in \mathbb{Q}$ is not countable

Question 2 (25 points)

2.1 (5 points). Here A and B denote propositions, each of which could be true or false. Prove or disprove the following proposition and clearly state whether the proposition is true or false.

$$((\neg A \implies B) \wedge (\neg A \implies \neg B)) \implies A$$

2.2 (5 points). Prove that among any set of n+1 integers one can find 2 numbers so that their difference is divisible by n.

2.3 (5 points). It's possible to run a version of the traditional propose & reject algorithm where the men and women switch roles (women propose, men say "maybe" or "yes", etc.). We'll refer to this as the non-traditional propose & reject algorithm. Prove the following statement:

In a stable marriage instance, there is exactly one stable pairing **if and only if** the pairing produced by the traditional propose & reject algorithm is the same as the pairing produced by the non-traditional propose & reject algorithm.

2.4 (10 points). Prove the following identity by induction:

$$\forall n \geq 0. \sum_{i=1}^{n+1} i * 2^i = n * 2^{n+2} + 2$$

a) State and prove the base case for the induction.

b) State the induction hypothesis.

c) Complete the proof by stating and proving the induction step.

Question 3 (15 points)

3.1 (1 points). If it exists, find the inverse of: 20 mod 79 and express your answer as the smallest valid non-negative integer.

3.2 (2 points). If it exists, find the inverse of: 5 mod 23 and express your answer as the smallest valid non-negative integer.

3.3 (1 points). What is $2^{2^{2006}}$ mod 3? Express your answer as the smallest valid non-negative integer.

3.4 (1 points). Calculate $2^{125} \mod 127$. Express your answer as the smallest valid non-negative integer.

3.5 (5 points). Suppose that p is prime. Let X be an integer uniformly distributed on $\{1, \ldots, p^n - 1\}$. What is the probability that X doesn't have an inverse modulo p^n?

3.6 (5 points). Suppose that p is prime. Let X be a geometric random variable with probability $1/2$ of a successful trial. What is the probability that X doesn't have an inverse modulo p^n? (hint: $\sum_{i=1}^{\infty} a^i = \frac{1}{1-a} - 1$ when $0 < a < 1$)

Question 4 (15 points) Some people just don't care

Let's suppose we have a stable marriage instance with n pairs of men and women. For whatever reason, everyone has decided that the other sex is pretty much the same to them, so rather than create a preference list by ranking, they will generate a list by ordering the other sex uniformly at random. Assume each person's list is independently generated.

4.1 Warm up (5 points). You do not need to show your work for these (they are worth 1 point each).

(a) How many different preference lists can a person generate?

(b) What is the probability that M_1 will have W_1 at the top of his list?

(c) What is the probability that TMA will terminate in exactly 1 day?

(d) What is the probability that TMA will terminate in exactly 1 day and produce a female-optimal pairing?

(e) What is the probability that every man has the same preference list?

4.2 (5 points). What is the probability that M_1 will prefer W_1 over W_2? Explain your answer.

4.3 (5 points) After randomly generating his preference list, M_1 suddenly realizes that he does have a preference for the women after all! However, he's still not sure what that preference is, and so he comes up with the following way to figure it out:

1. Order the pairs of positions on his list (note that this is a pair of **positions**, *not* women). Denote each position in a pair as r_1 and r_2.
   For each pair:
2. If he likes the woman currently listed at r_2 more than r_1, switch the women's places. Otherwise, leave them in their current positions.
3. Repeat until all pairs have been checked.

What is the expected number of switches he will have to make before he recovers his preference list? Assume he never changes his mind about two women.
[Hint: Each swap decreases the number of inverted pairs by 1.]

Question 5 (15 points) Caverns and Drakes

After much anticipation, the wonderful people at Hail® have released their newest RPG (role-playing game), *Caverns and Drakes*. Not wanting to fall behind your friends, who camped out to buy it, you quickly read up on the rules and decide to create a powerful fighter.

5.1 (5 points). Character creation works as follows: roll 4 fair six-sided dice, ignore the lowest roll (in the event of a tie, pick one to ignore), and total the remaining results (for example, if the dice resulted in 6, 5, 5, 3, the total would be $6 + 5 + 5 = 16$). Do this twice to determine your Attack and Defense ratings.

(a) **2 points.** Identify the probability space ω.

(b) **2 points.** What is the probability of getting an Attack rating of 18?

(c) **1 point.** What is the probability of getting an Attack rating of 3?

5.2 (5 points). Let's suppose that after you roll, your Attack and Defense are both 10. It's time to go out and kill things!

Combat works as follows: each round, roll a 20-sided die, add your Attack rating to the result (we will call this the *Attack Value*), and compare it to your target's Defense + 10 (the extra 10 on Defense is for balance issues–otherwise, everyone would be hitting everything!). If you exceed that number, you hit and deal damage. Otherwise, you miss (and deal no damage).
Each character (and monster) also has a Hit Point total, which damage is subtracted from. If your Hit Point total goes below 0, you die.

You must show your work–answers with no justification will receive *no* credit.

(a) **1 point.** A Troll has a Defense of 10. What is the probability that you will hit it with an attack?

(b) **1 point.** Suppose you have a greatsword, whose damage is calculated as follows: roll two 6-sided dice and add the results. Given that you hit the Troll, how much damage can you expect to deal?

(c) **1 point.** How much damage should you expect to deal to the Troll each round?

(d) **2 points.** If you encounter a Troll, should you fight it?
(A Troll has an Attack rating of 14 and 40 Hit Points. You have 50 Hit Points. The damage for a Troll's attack, its claw, is calculated as follows: roll a 6-sided die and add 6.5 to the result.)

5.3 (5 points). Later that night, you friend boasts about a rare bow that they found. It has the following special ability: each round, you may choose to attack twice (normally, you are only allowed to attack once per round), but both attacks suffer a -2 penalty when determining whether or not they hit (damage is unaffected).

For example, say you use this ability and get a 10 and a 12 on the two die rolls. Normally, the Attack Value for each result would be 20 and 22, respectively, but because of the -2 penalty, the final Attack Value for each attack would instead be 18 and 20, respectively.

(a) **3 points**. The bow's damage is calculated as follows: roll an 8-sided die and add 4 to the result. Your friend has an Attack rating of 9. Should they use the bow's ability while fighting the Troll?

(b) **2 points**. Suppose your friend goes on vacation, and you get to thinking about that special ability from (a). Let y be the probability that you will hit a given monster, and let x be the expected value of the damage you will deal if you hit (note that this is ***not*** the expected damage!). For what values of x and y is it advantageous to use that special ability?

You may assume you never do 0 damage, given that you hit.

Question 6 (15 points) I infer that I should scream GO Bears!

The valiant California Golden Bears are in the fight of their lives against the hated Stanford Cardinal. So far, the Bears have been completely unable to get any offensive production. At halftime, star quarterback Zach Maynard has a talk with his coach, and they come to the following conclusions:
Stanford's defense can be in one of two formations, Zone coverage or Man coverage. Because they don't know which formation Stanford will be aligned in to start the second half, they assume both are equally likely.
If Stanford is in Man coverage, Zach knows that he complete a pass with probability 0.8. However, if they are in Zone coverage, he can complete a pass with probability 0.3.
[For each part, you may define any quantity from a previous part as a variable and use it with no penalty.]

6.1 (5 points). On the opening drive of the second half, Zach is able to complete a pass. What is his new guess for the probability that Stanford is in Zone coverage?

6.2 (5 points). Through extensive film study, Zach Maynard knows that if the Stanford defense is in Zone coverage for a given play, they will stay in Zone coverage with probability $\frac{3}{4}$ the following play.

Given Zach's updated belief about Stanford's defense (that is, that Zach just completed a pass), what will be his guess for their formation on the following play?
You may assume (as many offensive coordinators do) that each play depends only on the previous play.
Hint: if an event A depends only on event B, then $\forall C.(Pr[A|B, C] = Pr[A|B])$.

6.3 (5 points) Suppose that Zach Maynard completes another pass on the following play. Given everything that has happened, what is his guess for the probability that Stanford is in Zone coverage now?

Question 7 (15 points)

CS70 teaching staff are trying to infer how many GSIs would need to be present at the final exam to be able to answer questions students might have on the exam. Each GSIs recorded the statistics of how often students enrolled in their sections ask questions during the last midterm exams. You can assume the students ask questions independently from each other. You can assume the number of questions students ask follows Poisson distribution.

- Hyun Oh's students ask N_1 number of questions per every 60 minutes on average.
- Victor's students ask N_2 number of questions per every 90 minutes on average.
- Dapo's students ask N_3 number of questions per every 30 minutes on average.

7.1 (7 points). What is the probability that students ask at least two questions for the three hour(180 min) final exam?

7.2 (8 points). If we let Z be the number of questions asked by students from one of the three sections with least number of questions, what is the probability that $Z \geq 2$?

Question 8 (15 points)

You and your best friend Ned Stark are watching your mutual enemy Jaime Lannister throw darts at a board. It is known that each time Jaime tosses a dart at the dartboard, the distances R_i from the bullseye can be modeled as a Uniform$(0, 1)$ random variable (the distance is uniform on (0,1), and the radius of the dartboard is not necessarily 1.)

Moreover, it is known that the tosses are independent.

You and Ned agree to the following wager

1. You will both observe Jaime toss 100 darts at a dart board, each a distance R_i from the bullseye.
2. If the sum of the distances is greater than or equal to 55, Ned wins $10.
3. If the sum is between $50 - \delta$ and $50 + \delta$, you win $1.
4. Otherwise, no one wins the wager.

Let X denote the sum of the the distances of the 100 dart tosses.

8.1 (1 points). Compute $\mu = E[X]$.

8.2 (2 points). Compute $V = Var[X]$.

8.3 (2 points). Using Markov's inequality, approximate the probability that Ned wins. Your answer should be expressed in terms of some combination of μ, V (note that your final answer might or might not use all of these variables.)

8.4 (2 points). Using Chebyshev's inequality, approximate the probability that Ned wins. Your answer should be expressed in terms of some combination of μ, V (note that your final answer might or might not use all of these variables.)

8.5 (3 points). Let the function $\Phi(x) = \Pr(Z \leq x)$, where Z is a $N(0,1)$ random variable. Approximate the probability that Ned wins using the central limit theorem (your final answer should involve $\Phi(a)$ for some number a.) Your answer should be expressed in terms of some combination of μ, V (note that your final answer might or might not use all of these variables.)

8.6 (5 points). For the game to be fair, your expected payoff and that of Ned should be equal. We will now calculate what δ must be for the game to be fair. In other words, if θ_n denotes the probability of Ned winning and θ_u denotes your probability of winning, then $10\theta_n = \theta_u$.

Assume that θ_n is 0.05 (this may or may not be the answer to previous parts of the problem, don't necessarily assume that it is, or use those previous calculated values of θ_n for this part.)

Using the central limit theorem, (approximately) choose δ so that the game is fair.

You will find it useful to use $\Phi^{-1}(x)$ to denote the inverse of the function $\Phi(x)$. In other words, given any x, if $y = \Phi(x)$, then $\Phi^{-1}(y) = x$.