CS 70 Second Midterm 26 July 2012

NAME (1 pt): ______________________________

SID (1 pt): ______________________________

TA (1 pt): ______________________________

Name of Neighbor to your left (1 pt): ______________________________

Name of Neighbor to your right (1 pt): ______________________________

Instructions: This is a closed book, closed calculator, closed computer, closed network, open brain exam, but you are permitted a 1 page, double-sided set of notes, large enough to read without a magnifying glass.

You get one point each for filling in the 5 lines at the top of this page.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

1	
2	
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4	
5	
Total	

1

Question 1 (20 points) True or False.

For each of the following propositions, circle either T if it is always true, F if it is always false. You do not have to justify your answer. A and B denote events in the sample space S.

T F If A and B are disjoint then A and B are independent

Answer: *False*

T F $\Pr(\overline{A} \cap \overline{B}) = 1 - \Pr(A) - \Pr(B) + \Pr(A \cap B)$

Answer: *True*

T F Let n_1, n_2, n_3 be the number of students enrolled in Hyun Oh's, Victor's and Dapo's section, and let x, y, z be the average homework scores for each section, respectively. The homework score averaged across all three sections is $\frac{x+y+z}{3}$.

Answer: *False*

Different sections were averaged on different number of students. So the class average is

$$\frac{n_1 x + n_2 y + n_3 z}{n_1 + n_2 + n_3}.$$

T F Let X, Y be random variables. Furthermore, X is a 0/1 valued indicator random variable. Then, the following holds: $\sum_{a \in \mathcal{A}} \Pr(X = 1 | Y = a) = 1$.

Answer: *False*

If X and Y are independent, then $\sum_{a \in \mathcal{A}} \Pr(X = 1 | Y = a) = \Pr(X = 1)$ which doesn't necessarily equal to 1.

T F If the random variables X and Y are independent, and $X \sim Bin(n, p)$ and $Y \sim Bin(m, p)$, then the random variable $Z = X + Y \sim Bin(n + m, p)$

Answer: *True*

We need to show that $\Pr(Z = z) = \binom{n+m}{z} p^z (1-p)^{n+m-z}$

$$
\begin{aligned}
\Pr(Z = z) &= \sum_{i=0}^{z} \Pr(X = i \cap Y = z - i) \\
&= \sum_{i=0}^{z} \Pr(X = i) * \Pr(Y = z - i) \\
&= \sum_{i=0}^{z} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{z-i} p^{z-i} (1-p)^{m-(z-i)} \\
&= \sum_{i=0}^{z} \binom{n}{i} \binom{m}{z-i} p^i p^{z-i} (1-p)^(n-i)(1-p)^{m-(z-i)} \\
&= \sum_{i=0}^{z} \binom{n}{i} \binom{m}{z-i} p^z (1-p)^{n+m-z} \\
&= p^z (1-p)^{n+m-z} \sum_{i=0}^{z} \binom{n}{i} \binom{m}{z-i} \\
&= p^z (1-p)^{n+m-z} \binom{n+m}{z}
\end{aligned}
$$

T F $A - B$ and $A \cap B$ are disjoint for any two events A and B over a probability space.

Answer: *True*

T F $\Pr(A|B) = \Pr(A)$ if and only if A and B are independent.

Answer: *True*
$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A)$.

T F For any three events A, B, C, if $\Pr(A \cap B) > 0$ and $\Pr(B \cap C) > 0$, then $\Pr(A \cap C) > 0$.

Answer: *False*

Let $\Omega = \{1, 2, \ldots 100\}$, with $\Pr(x) = \frac{1}{100}$ for every $x \in \Omega$. Let B be the odd numbers in Ω, C be the even numbers in Ω, and $A = \{1, 2, \ldots, 10\}$.

T F Given two events A and B with $\Pr(A) > 0$ and $\Pr(B) > 0$, if $\Pr(A|B) > \Pr(A)$, then $\Pr(B|A) > \Pr(B)$.

Answer: *True*

Essentially:

$$
\begin{aligned}
\Pr(B|A) &= \frac{\Pr(B \cap A)}{\Pr(A)} \text{ (Bayes rule)} \\
&= \frac{\Pr(A|B)\Pr(B)}{\Pr(A)} \text{ (Bayes rule again)} \\
&> \frac{\Pr(A)\Pr(B)}{\Pr(A)} \text{ (given to us in problem)} \\
&= \Pr(B).
\end{aligned}
$$

T F Two cards are drawn at random without replacement from a standard deck of 52 playing cards (In other words, the first card is drawn at random, then the second card is drawn at random from the remaining cards.) Then the event that the two cards are both aces is independent of the event that they are both diamonds.

Answer: *False*
Let $A = \{\text{I drew two aces}\}$, $D = \{\text{I drew two diamonds}\}$. Then

$$
\begin{aligned}
\Pr(A) &= \frac{4 \times 3}{52 \times 51} \\
\Pr(D) &= \frac{13 \times 12}{12 \times 51},
\end{aligned}
$$

but $\Pr(A \cap D) = 0$, since the diamonds suit only has 1 ace card.

Question 2 (20 points)

2.1 (5 points). You and your friend start working at the same company. You and your friend are randomly assigned one of 6 desks in a row. What is the probability that you do not sit at adjacent desks?

Answer: Let's count the number of ways we could sit at adjacent desks. Assume we sit together, there are 5 places we could sit and 2 permutations = 10 possible ways.
In total, there are 6*5=30 possible ways we could be assigned desks.
So there $Pr(\text{not adjacent}) = 1 - (10/30) = 2/3$

2.2 (5 points). Using a combinatorial argument, show that for all nonnegative integers n and k and r with $k \leq r \leq n$

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

Answer: Suppose we want to choose an r person committee and a k person subcommittee. We can chose all the people in the committee and then select the subcommittee from that group. This is the same as choosing the subcommittee from all people and then choosing the remaining members of the committee. In the end, we have 3 groups. The $n - r$ people not in any committee, the $r - k$ people in the committee but not in the subcommittee, and the k people in the subcommittee.

2.3 (5 points). Suppose I roll a weighted d-sided die n times, where each side i has probability p_i of being rolled. What is the probability of rolling k 1's? How many 1's can I expect?

Answer: This is binomial, since we don't care how many of the other die results we get:

$$\Pr(\text{getting } k \text{ 1's}) = \binom{n}{k} p_1^k (1-p_1)^{n-k}$$

Similarly, $\mathbb{E}(X_1) = np_1$, where X_i is a random variable denoting the number of i's that appear in n die rolls.

2.4 (5 points). How many solutions does $z_1 + z_2 + z_3 + 4 * z_4 = 11$ where each z_i is a non-negative integer?

Answer: This is close to a stars and bars problem, but the $4 * z_4$ forces us to consider cases . We could either count solutions to $z_1 + z_2 + z_3 + z_4 = 11$ and subtract the ones where z_4 is divisible by 4 or divide it into cases on the 3 possible values for z_4.
When $z_4 = 0$ there are 11 stars, 2 bars and $\binom{13}{2}$ solutions
When $z_4 = 1$ there are 7 stars, 2 bars and $\binom{9}{2}$ solutions
When $z_4 = 2$ there are 3 stars, 2 bars and $\binom{5}{2}$ solutions
So there are $\binom{13}{2} + \binom{9}{2} + \binom{5}{2}$ total possible solutions

Question 3 (20 points)

Ziegfried (a friend of yours who has not taken CS70) suggests the following dice game:

1. Two dice are repeatedly tossed until the sum is 10 (in which case Ziegfried wins) or the sum is 7 (in which case you win).
2. Once a 10 or a 7 is tossed, the game is over.

Upon hearing the rules of this game, you quickly agree to play.

Let p denote the probability that the outcome on any given toss is 10, q denote the probability that the outcome on any given toss is 7 (of course, $p + q$ does not necessarily equal one, the game might not end on a given toss!)

3.1 (1 points). Suppose that $p < q$. What is your intuition for who is more likely to win the game? Ziegfried (associated with p) or you (associated with q)? Write and box your answer. No explanation is needed.

Answer: The player associated with q (You) is more likely to win.

3.2 (1 points). On the first toss, what is the probability that the sum is 10? In other words, compute p.

Answer: (4,6), (5,5), (6,4) are the outcomes that result in the sum 10. Each outcome has probability $\frac{1}{6^2}$. So, $\frac{3}{6^2} = \frac{1}{12}$.

3.3 (1 points). On the first toss, what is the probability that the sum is 7? In other words, compute q.

Answer: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) are the outcomes that result in the sum 7. Each outcome has probability $\frac{1}{6^2}$. So, $\frac{1}{6}$.

3.4 (1 points). What is the probability that the game is completed in exactly 3 tosses?

Answer: The probability of rolling a 10 or a 7 on a trial is $\frac{1}{12} + \frac{1}{6} = \frac{1}{4}$. So, the desired probability is $(\frac{3}{4})^2 \frac{1}{4}$.

3.5 (4 points). What is the probability that the game is completed in 3 or less tosses?

Answer: If we let N denote the trial number for which the game ends, then this question asks $\Pr(N \leq 3)$. Let X_i denote the outcome of the i^{th} toss. Define the event $D_i = \{X_i \neq 7\} \cap \{X_i \neq 10\}$

We compute

$$\Pr(N > 3) = \Pr(\bigcap_{i=1}^{3} D_i) = (\frac{3}{4})^3$$

So $1 - (\frac{3}{4})^3$.

3.6 (4 points). Suppose that we know that the game ends in 3 tosses. What is the probability that Ziegfried wins?

Answer: Let $B = \{\text{Game ends in 3 tosses}\}$. Let $A_i = \{\text{winning toss is a } i\}$. Define $C_7 = B \cap A_7$ and $C_{10} = B \cap A_{10}$.

$$
\begin{aligned}
\Pr(\text{Ziegfried wins}|B) &= \Pr(C_{10}|B) \\
&= \frac{\Pr(C_{10}\cap B)}{\Pr(B)} = \frac{\Pr(C_{10})}{\Pr(B)} = \frac{(\frac{3}{4})^2\frac{1}{12}}{(\frac{3}{4})^2[\frac{1}{12}+\frac{1}{6}]} = \frac{1}{3}
\end{aligned}
$$

3.7 (8 points). Now, suppose that we don't know when the game will end. What is the probability that Ziegfried wins?

(Hint: You may find the following law of total probability useful: when $A = \cup_{i=1}^{\infty} A_i$ and $A_i \cap A_j = \emptyset \forall i \neq j$, then $\Pr(B \cap A) = \Pr(\cup_{i=1}^{\infty} \{B \cap A_i\}) = \sum_{i=1}^{\infty} \Pr(B \cap A_i)$.)

Answer: If you look closely at the computation done in 3.6, notice that it can be generalized to arbitrary n, allowing us to conclude that

$$
\forall n \in \mathbb{N}, \Pr(\text{Ziegfried wins}|\text{ last trial is } N = n) = \frac{1}{3}. \tag{1}
$$

Using the law of total probability (LTP), Bayes rule (BR) and equation (1), we conclude that

$$
\begin{aligned}
\Pr(\text{Ziegfried wins}) &= \Pr(\cup_{i=1}^{\infty} \{\text{Ziegfried wins on toss } i\}) \\
&= \sum_{i=1}^{\infty} \Pr(\{\text{Ziegfried wins on toss } i\}) \text{ (by the LTP)} \\
&= \sum_{n=1}^{\infty} \Pr(\text{Ziegfried wins}|\text{ last trial is } N = n) \Pr(\text{ last trial is } N = n) \text{ (by BR)} \\
&= \sum_{n=1}^{\infty} \frac{1}{3} \Pr(\text{ last trial is } N = n) \text{ (by equation (1))} \\
&= \frac{1}{3} \sum_{n=1}^{\infty} \Pr(\text{ last trial is } N = n) \\
&= \frac{1}{3} \Pr(\bigcup_{n=1}^{\infty} \{\text{ last trial is } N = n\}) \text{ (LTP)} \\
&= \frac{1}{3} \Pr(\Omega) \\
&= \frac{1}{3}.
\end{aligned}
$$

This formally justifies 3.1.

Question 4 (20 points) There was an outbreak of Mad Squirrel Disease and you're asked to help the biologists at CDC with probabilistic analysis. They have captured n random squirrels out on Hearst Avenue and put them in a cage for in-lab experiments. The experiments showed that infected squirrels have red eyes with probability r and healthy squirrels have red eyes with probability s. Also the probability a squirrel is infected with the disease is p.

4.1 (3 points). What is the probability that a randomly chosen squirrel from the cage is healthy and doesn't have red eyes?

Answer:

$$\Pr(\text{healthy}, \neg\text{red eyes}) = \Pr(\neg\text{red eyes}|\text{healthy})\Pr(\text{healthy}) = (1-s)(1-p)$$

A clumsy lab assistant from another lab captured x additional infected squirrel with red eyes and accidentally put them into our cage. So there must be $n+x$ squirrels in the cage now. Our experiment log shows that there were t number of infected squirrels before the lab assistant mix up the old and the new squirrels. (Note: $x < t < n$)

4.2 (8 points). What is the probability that a randomly chosen squirrel after the accident has red eyes?

Answer:

$$\begin{aligned}
&\Pr(\text{red eyes}) = \\
&\Pr(\text{red eyes} \mid \text{new infected squirrels with red eyes})\Pr(\text{new infected squirrels with red eyes}) \\
&+ \Pr(\text{red eyes} \mid \text{old infected squirrels})\Pr(\text{old infected squirrels}) \\
&+ \Pr(\text{red eyes} \mid \text{old healthy squirrels})\Pr(\text{old healthy squirrels}) \\
&= 1\frac{x}{n+x} + r\frac{t}{n+x} + s\frac{n-t}{n+x} = \frac{x+rt+s(n-t)}{n+x}
\end{aligned}$$

4.3 (9 points). What is the probability that a randomly chosen squirrel after the accident is infected given that it has red eyes?

Answer:

$$\Pr(\text{infected} \mid \text{red eyes}) = \frac{\Pr(\text{red eyes} \cap \text{infected})}{\Pr(\text{red eyes})}$$

$$\begin{aligned}
&\Pr(\text{red eyes} \cap \text{infected}) = \\
&\Pr(\text{red eyes} \cap \text{infected} \mid \text{new infected squirrels with red eyes}) \times \\
&\qquad\qquad\qquad \Pr(\text{new infected squirrels with red eyes}) \\
&+ \Pr(\text{red eyes} \cap \text{infected} \mid \text{old infected squirrels}) \Pr(\text{old infected squirrels}) \\
&+ \Pr(\text{red eyes} \cap \text{infected} \mid \text{old healthy squirrels}) \Pr(\text{old healthy squirrels}) \\
&= 1\frac{x}{n+x} + r\frac{t}{n+x} + 0\frac{t-x}{n+x} = \frac{x+rt}{n+x}
\end{aligned}$$

Pr(red eyes) was computed in the previous part.

$$\Pr(\text{infected} \mid \text{red eyes}) = \frac{\frac{x+rt}{n+x}}{\frac{x+rt+s(n-t)}{n+x}} = \frac{x+rt}{x+rt+s(n-t)}$$

Question 5 (15 points) The Claaaaaawwww!!!

Your mom recently decided to take you to *Pizza Planet*, a space-themed restaurant based on your favorite action figure, *Breeze Leet-ear*. You notice that the place has a claw machine with a lot of neat toys, and decide to try it out.

5.1 (10 points). Suppose there are n toys in the machine, and the claw can grab any number of them on a given try (formally, on a given attempt, the claw can grab k toys, $k \in \{1, 2, \ldots, n\}$). Furthermore, suppose the claw can grab each *set* of toys with equal probability (that is, the machine is equally likely to grab 1 toy or all the toys).
We will call the set of items grabbed I, where $I \subseteq \{1, 2, \ldots, n\}$ and $|I| = k$.

(i) For a given I and object i, what is the probability that $i \in I$?

Answer:

$$\begin{aligned}
\Pr(i \in I) &= \frac{|\text{\# of ways to pick } I \text{ containing } i|}{|\text{total \# of ways to pick } I|} \\
&= \frac{\binom{n-1}{k-1}}{\binom{n}{k}} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{k!(n-k)!}{n!} \\
&= \frac{k}{n}
\end{aligned}$$

Answer 2: We could compute the numerator by taking the complement instead:

$$\begin{aligned}
\Pr(i \in I) &= \frac{|\text{\# of ways to pick } I \text{ containing } i|}{|\text{total \# of ways to pick } I|} \\
&= \frac{\binom{n}{k} - \binom{n-1}{k}}{\binom{n}{k}}
\end{aligned}$$

(Check that this gives the same result!)

(ii) Suppose for each item i, the manager has attached a price, $v(i)$. On average, how much swag will you get (ie what is $\mathbb{E}(I)$)?
Express your answers in terms of k, n, $v(i)$, or I (inclusive or).

Answer:

$$\begin{aligned}
\mathbb{E}(I) &= \sum_{i=1}^{n} \Pr(i \in I) \cdot v(i) \\
&= \frac{k}{n} \times \sum_{i=1}^{n} v(i)
\end{aligned}$$

5.2 (5 points). Unbeknownst to you, your two favorite toys found their way into this machine, and were captured by your neighbor (to be strapped to fireworks and exploded! Oh dear). Upset by losing so much value, the store owner changed the claw's behavior so that it now always grabs exactly $\frac{n}{4}$ toys (assume n is divisible by 4).

If the sum total of the toys in the machine is \$20, how much should the manager charge per game so that, on average, he makes money?
[You may define any quantity from 5.1 as a variable and use it with no penalty]

Answer: From 5.1(ii), we know $\mathbb{E}(I) = \frac{k}{n} \times \sum_{i=1}^{n} v(i)$. Now, we're told that $k = \frac{n}{4}$ and that $\sum_{i=1}^{n} v(i) = 20$. Substituting, we have

$$\begin{aligned}
\mathbb{E}(I) &= \frac{k}{n} \times \sum_{i=1}^{n} v(i) \\
&= \frac{\frac{n}{4}}{n} \times 20 \\
&= \frac{20}{4} = 5
\end{aligned}$$

Therefore, the manager should charge more than \$5 per game for it to be profitable.