EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 14 February 2012
University of California Berkeley

LAST Name Delta FIRST Name Dee

Lab Time High Noon

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" $\times$ 11" sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 6.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the six numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (30 Points) The input-output behavior of a continuous-time system is described by the following equation:

$$y(t) = \begin{cases} 0 & \text{if } x(t) \le 0 \\ x(t) & \text{if } x(t) > 0, \end{cases}$$

where x denotes the input and y the output. For each part, explain your reasoning succinctly, but clearly and convincingly.

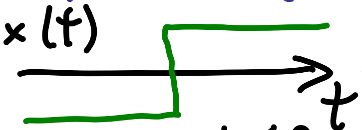
(a) (15 Points) Select the strongest true assertion from the list below.

(i) The system must be linear.

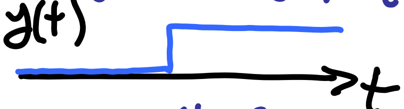
(ii) The system could be linear, but does not have to be.

$\boxed{\text{(iii) The system cannot be linear.}}$ (*)

Many ways to show this. For example, consider the input $x(t) = \text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$



. The corresponding output is



$y(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}$. However, the input $\hat{x}(t) = -x(t)$ for all t produces $\hat{y}(t) = \begin{cases} 1 & t < 0 \\ 0 & t \ge 0 \end{cases} \ne -y(t)$.

(b) (15 Points) Select the strongest true assertion from the list below.

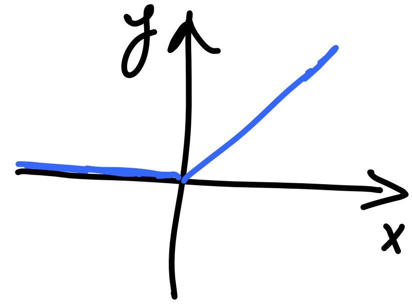
$\boxed{\text{(i) The system must be time invariant.}}$

(ii) The system could be time invariant, but does not have to be.

(iii) The system cannot be time invariant.

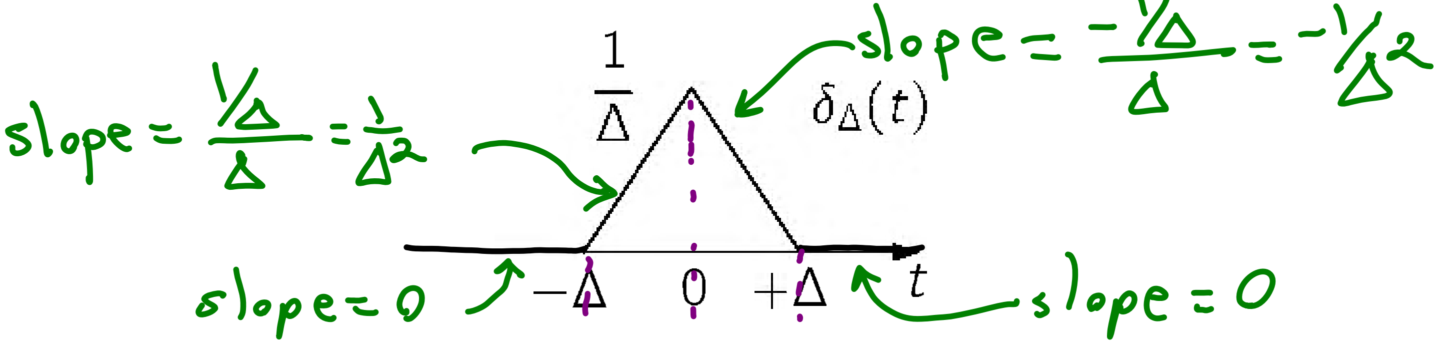
Let $\hat{x}(t) = x(t-T) \Rightarrow \hat{y}(t) = \begin{cases} 0 & \text{if } \hat{x}(t) \le 0 \\ \hat{x}(t) & \text{if } \hat{x}(t) > 0 \end{cases} = \begin{cases} 0 & \text{if } x(t-T) \le 0 \\ x(t-T) & \text{if } x(t-T) > 0 \end{cases}$

$= y(t-T) \quad \forall t, T$



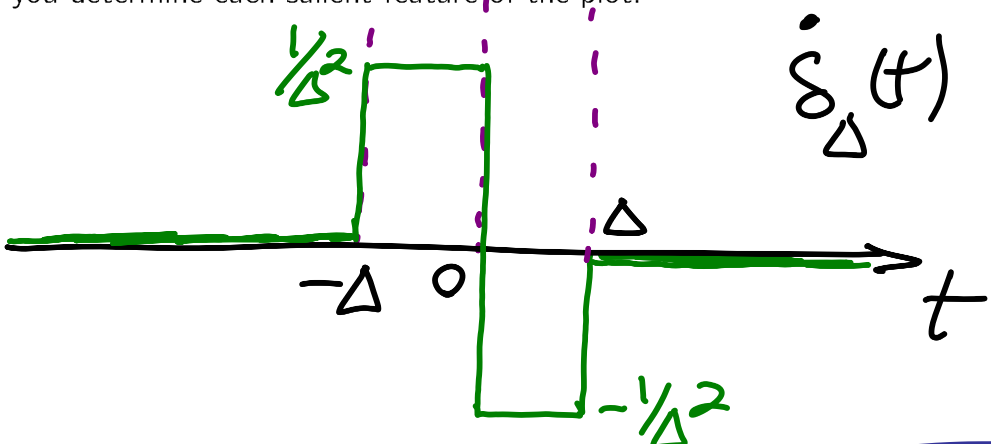
(*) This system is "piecewise linear" similar to an ideal diode.

MT1.2 (55 Points) Consider the continuous-time signal δ_Δ shown below.

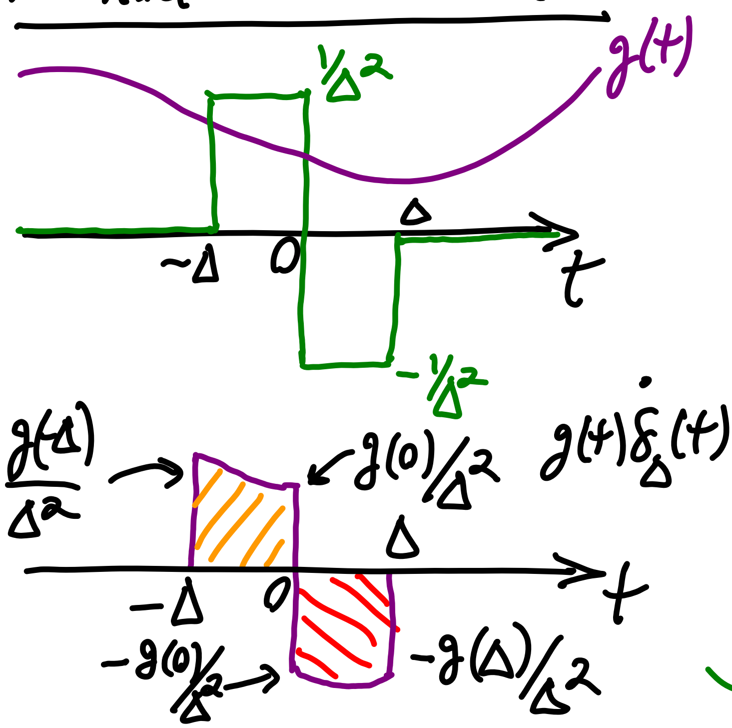


We know that in the limit $\Delta \to 0$, the triangular pulse behaves like a unit-strength Dirac delta. In this problem, you will explore the behavior of $\dot{\delta}_\Delta(t) \triangleq d\delta_\Delta(t)/dt$.

(a) (10 Points) Provide a well-labeled plot of $\dot{\delta}_\Delta(t)$; you must explain briefly how you determine each salient feature of the plot.



Method 1 for Part (b):



$$\int g(t)\dot{\delta}_\Delta(t)\,dt = \boxed{\quad} + \boxed{\quad}$$

By the Trapezoidal rule,

$$\boxed{\quad} \approx \frac{g(-\Delta)+g(0)}{\Delta^{\cancel{2}}}\,\frac{\cancel{\Delta}}{2}$$

$$\boxed{\quad} \approx -\frac{g(0)+g(\Delta)}{\Delta^{\cancel{2}}}\,\frac{\cancel{\Delta}}{2}$$

$$\boxed{\quad} + \boxed{\quad} \approx -\frac{g(\Delta)-g(-\Delta)}{2} \approx -\dot{g}(0)$$

$$\int g(t)\dot{\delta}_\Delta(t)\,dt \approx -\dot{g}(0)$$

(b) (15 Points) Consider a smooth test function g. Provide a reasonable approximation (in terms of g) to the following integral:

$$\int_{-\infty}^{+\infty} g(t)\,\dot{\delta}_\Delta(t)\,dt.$$

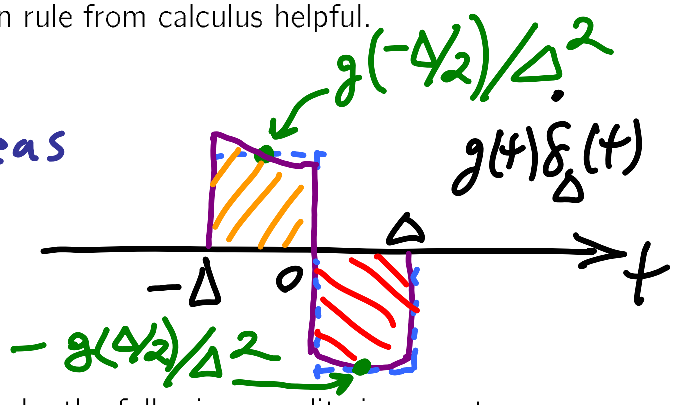
You may find the trapezoidal approximation rule from calculus helpful.

Method 2:

$\int g(t)\,\dot{\delta}_\Delta(t)\,dt \approx$ sum of rectangular areas defined by midpoints

$$= \frac{g(-\Delta/2)}{\Delta^2}\Delta - \frac{g(\Delta/2)}{\Delta^2}\Delta$$

$$\approx -\dot{g}(0)$$



(c) (15 Points) Let $\dot{\delta}(t) = \lim_{\Delta\to 0} \dot{\delta}_\Delta(t)$. Explain why the following equality is correct:

$$\int_{-\infty}^{+\infty} g(t)\,\dot{\delta}(t)\,dt = -\dot{g}(0).$$

The approximation by the Trapezoidal or rectangular areas becomes exact in the limit $\Delta \to 0$.

Note that $\dot{\delta}$ is an odd function (as is clear from $\dot{\delta}_\Delta$): $\dot{\delta}(-t) = -\dot{\delta}(t)$

(d) (15 Points) Use the result of part (c) to evaluate the integral

$$\int_{-\infty}^{+\infty} g(\tau)\,\dot{\delta}(t-\tau)\,d\tau. = (g \star \dot{\delta})(t)$$

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$$g(t) \longrightarrow \boxed{\dot{\delta}(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} g(\tau)\,\dot{\delta}(t-\tau)\,d\tau = ?$$

Note that $y(0) = \int_{-\infty}^{\infty} g(\tau)\,\dot{\delta}(-\tau)\,d\tau = -\int_{-\infty}^{\infty} g(\tau)\,\dot{\delta}(\tau)\,d\tau = -(-\dot{g}(0)) = \dot{g}(0)$

We conjecture that $y(t) = \dot{g}(t)$, and this turns out to be correct.

Rewrite the convolution as $y(t) = \int_{-\infty}^{\infty} g(t-\mu)\,\dot{\delta}(\mu)\,d\mu$, and we see that

$$y(t) = -\frac{d}{d\mu} g(t-\mu)\Big|_{\mu=0} = -\left(\left(\frac{du}{d\mu}\Big|_{\mu=0}\right)\left(\frac{dg(u)}{du}\Big|_{u=t-\mu|_{\mu=0}=t}\right)\right) = \dot{g}(t)$$

let $u = t-\mu \Rightarrow du/d\mu = -1$

chain rule

MT1.3 (20 Points) Determine and provide a well-labeled plot containing each of the 4^{th} roots of $z = \dfrac{16}{\sqrt{2}} + i\dfrac{16}{\sqrt{2}}$; be sure to draw the unit circle so the positions of the roots relative to it are clear. Also, determine the sum of all the fourth roots.

Using the fact that

$$\cos(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$$

write

$$z = 16\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 16\, e^{i\pi/4}$$

To draw the fourth roots, rewrite z as follows:

$$z = 16\, e^{i\left(\frac{\pi}{4} + 2\pi k\right)} \qquad k \in \mathbb{Z}$$

Then

$$z^{1/4} = 16^{1/4}\, e^{i\left(\frac{\pi}{16} + \frac{\pi}{2}k\right)} = 2\, e^{i\left(\frac{\pi}{16} + \frac{\pi}{2}k\right)}$$

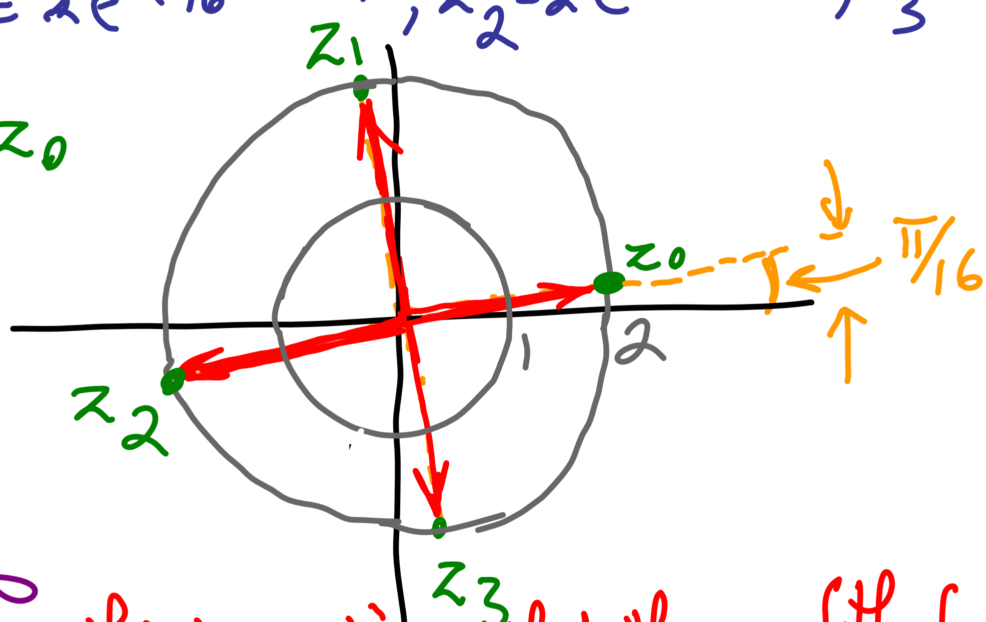
All the fourth roots of z are on the circle of radius 2, and at locations determined by $k = 0, 1, 2, 3$

$$z_0 = 2e^{i\pi/16}, \quad z_1 = 2e^{i\left(\frac{\pi}{16} + \pi/2\right)}, \quad z_2 = 2e^{i(\pi/16 + \pi)}, \quad z_3 = 2e^{i\left(\frac{\pi}{16} + \frac{3\pi}{2}\right)}$$

Clearly, $z_2 = -z_0$ and $z_3 = -z_1$

$$\Downarrow$$

$$\sum_{k=0}^{3} z_k = 0$$



You can also see this by noticing that the sum of the four vectors is zero.

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Problem	Points	Your Score
Name	10	10
1	30	30
2	55	55
3	20	20
Total	**115**	115