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Math110 Midterm I, Fall 2011

   This is a closed book exam; but everyone is allowed a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so. Hand in this exam before you leave.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Write your personal information below and on top of every page in the test.

Your Name: _____

Your GSI: _____

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1. Let $\mathcal{P}$ be the set of all polynomials with real coefficients.

   (a) Let $\mathcal{Q}$ be the set of all polynomials of non-negative coefficients. Is $\mathcal{Q}$ a subspace of $\mathcal{P}$?
   Solution: No. Both $f(x) = 0$ and $g(x) = x$ are in $\mathcal{Q}$. But there is no additive inverse of $g(x)$ in $\mathcal{Q}$.

   (b) Let $\mathcal{R} = \{a_0 + a_1 x^2 + \cdots + a_n x^{2n} \mid \quad n \geq 0 \text{ integer}, a_0, a_1, \cdots, a_n \text{ are real numbers.}\}$. Is $\mathcal{R}$ a subspace of $\mathcal{P}$?
   Solution: Yes. $f(x) = 0$ is in $\mathcal{R}$. Let $f(x)$ and $g(x)$ be in $\mathcal{R}$. Both $f(x)$ and $g(x)$ are polynomials with only even terms. Then their sum is a polynomial with only even terms and must be in $\mathcal{R}$. Similarly let c be a real number, then $cf(x)$ is a polynomial with only even terms and must be in $\mathcal{R}$.

2. Determine whether the functions $\sin(x + 1)$, $\sin(x)$ and $\cos(x)$ are linearly dependent.
   Solution: Yes. $\sin(x + 1)$ is a linear combination of $\sin(x)$ and $\cos(x)$:

$$\sin(x + 1) = \cos(1)\sin(x) + \sin(1)\cos(x).$$

3. Let
$$S = \{(1, 2, 3), (2, 3, 4), (3, 4, 5)\}.$$

Find the dimension and a basis for **span**(S).

Solution: The vectors $(1, 2, 3), (2, 3, 4), (3, 4, 5)$ are linearly dependent:

$$2 * (2, 3, 4) - (3, 4, 5) - (1, 2, 3) = (0, 0, 0).$$

But vectors $(3, 4, 5)$ and $(1, 2, 3)$ are not multiples of each other and therefore linearly independent. Thus, the dimension of **span**(S) is 2, with $(3, 4, 5)$ and $(1, 2, 3)$ as a basis.

4. Define a function $T : \mathbf{R}^2 \to \mathbf{R}^4$ as

$$T\,(\,a_1, a_2\,) = (\,a_1 \quad 2a_1 \quad 3a_1 \quad 4a_1\,)\,.$$

(a) Show that T is a linear transformation.
   Solution: Let c be a real constant and $(\,a_1, a_2\,)$ and $(\,b_1, b_2\,)$ be two vectors of $\mathbf{R}^2$.
   Then

$$
\begin{aligned}
T\,((\,a_1, a_2\,) + c\,(\,b_1, b_2\,)) \;&=\; T\,(\,a_1 + cb_1, a_2 + cb_2\,) \\
&=\; (\,a_1 + cb_1 \quad 2(a_1 + cb_1) \quad 3(a_1 + cb_1) \quad 4(a_1 + cb1)\,) \\
&=\; (\,a_1 \quad 2a_1 \quad 3a_1 \quad 4a_1\,) + c\,(\,b_1 \quad 2b_1 \quad 3c \quad 4b1\,) \\
&=\; T\,(\,a_1, a_2\,) + cT\,(\,b_1, b_2\,)\,.
\end{aligned}
$$

(b) Find bases for its range and null spaces.
   Solution: Since
$$T\,(\,a_1, a_2\,) = a_1\,(\,1 \quad 2 \quad 3 \quad 4\,)\,.$$
   It follows that its range is **span**$\{(\,1 \quad 2 \quad 3 \quad 4\,)\}$, with dimension 1 and basis $(\,1 \quad 2 \quad 3 \quad 4\,)$.
   Let $T\,(\,a_1, a_2\,) = 0$. This implies $a_1 = 0$. So the null space is **span**$\{(0, 1)\}$, with dimension 1 and basis $(0, 1)$.

5. Let $g(x) = 1 + x$. Let $\mathbf{T} : \mathbf{P_2}(\mathbf{R}) \to \mathbf{P_2}(\mathbf{R})$ and $\mathbf{U} : \mathbf{P_2}(\mathbf{R}) \to \mathbf{R^3}$ be the linear transformations, respectively, defined by

$$\mathbf{T}(f(x)) = f'(x)g(x) + f(x), \quad \text{and} \quad \mathbf{U}(a + bx + cx^2) = (a + b, b + c, c + a).$$

Let β and γ be the standard ordered bases of $\mathbf{P_2}(\mathbf{R})$ and $\mathbf{R^3}$, respectively.

(a) Compute $[\mathbf{U}]_\beta^\gamma$, $[\mathbf{T}]_\beta$ and $[\mathbf{UT}]_\beta^\gamma$.
   Solution: Let $f(x) = a + bx + cx^2$. Then

$$\mathbf{T}(f(x)) = a + b + (2c + 2b)x + 3cx^2, \quad \mathbf{U}(\mathbf{T}(f(x))) = (a + 3b + 2c, 2b + 5c, a + b + 3c).$$

So
$$[\mathbf{U}]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad [\mathbf{T}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad [\mathbf{UT}]_\beta^\gamma = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{pmatrix}.$$

(b) Let $h(x) = 1 + 2x + 3x^2$. Compute $[h(x)]_\beta$ and $[\mathbf{U}(h(x))]_\gamma$.
   Solution:
$$[h(x)]_\beta = (1, \quad 2, \quad 3), \quad [\mathbf{U}(h(x))]_\gamma = (3, \quad 5, \quad 4).$$