University of California, Berkeley
Fall Semester 2009

College of Engineering
Professors R. Horowitz and A. Packard

E7 Midterm Examination II

Friday November 20, 2009

Name :	
SID :	

Section: **1** **2** (Please circle your lecture section)

Please mark your Laboratory section: (where your exam will be returned)

☐ `11: TuTh 8-10 (1109 Etch)` ☐ `12: TuTh 10-12 (1109 Etch)` ☐ `13: TuTh 12-2 (1109 Etch)`
☐ `14: TuTh 2-4 (1109 Etch)` ☐ `15: TuTh 4-6 (1109 Etch)` ☐ `16: MW 8-10 (1109 Etch)`
☐ `17: MW 10-12 (1109 Etch)` ☐ `18: MW 2-4 (1109 Etch)` ☐ `19: MW 4-6 (1109 Etch)`
☐ `20: MW 2-4 (2109 Etch)` ☐ `21: TuTh 10-12 (212 Wheeler)` ☐ `22: TuTh 12-2 (2109 Etch)`
☐ `23: TuTh 8-10 (212 Wheeler)` ☐ `24: MW 3-5 (212 Wheeler)`

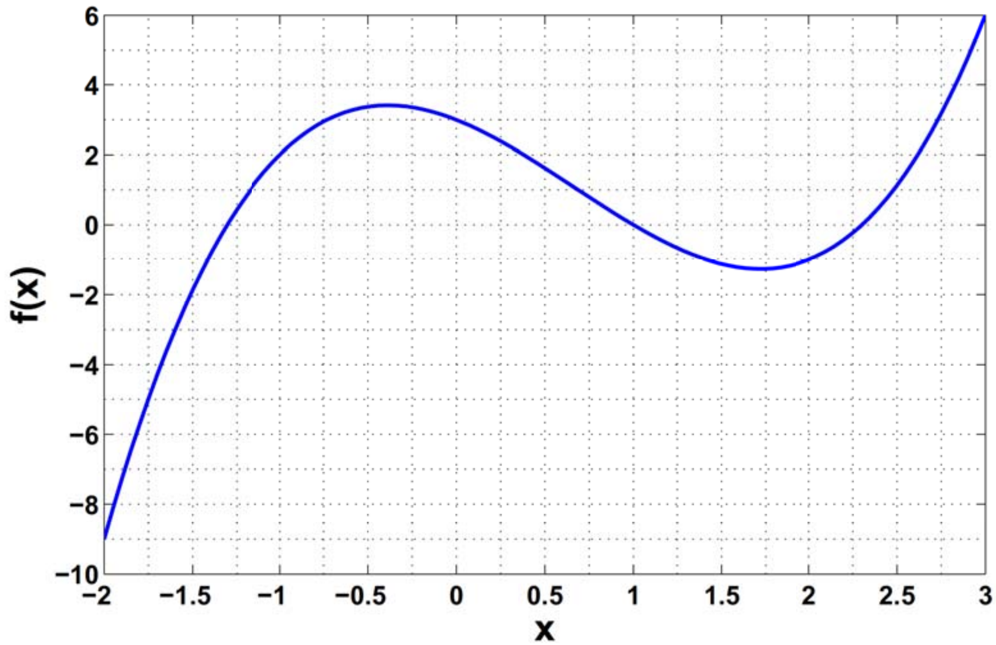
Part	Points	Grade
1	12	
2	10	
3	20	
4	16	
5	12	
6	18	
7	12	
TOTAL	100	

1. Write your name on each page.
2. Record your answers ONLY on the spaces provided.
3. You may not ask questions during the examination nor leave the room before the exam ends.
4. Close book exam. Two $8.5" \times 11"$ sheets of handwritten notes allowed.
5. No calculators or cell phones allowed. (Please turn cell phones off)

1. The graph of the polynomial function

$$f(x) = x^3 - 2x^2 - 2x + 3$$

is shown below. Its derivative is $\frac{d}{dx}f(x) = 3x^2 - 4x - 2$.



(a) Assume that you are using the bisection algorithm to find a root of $f(x)$ and that the initial search interval is $[x_L, x_R] = [-2, 3]$. In the space provided below write down the value of the root that the algorithm will find, by reading its approximate value from the graph [1].

`Ans:`

(b) Assume that you are using the Newton-Raphson algorithm and that the initial root estimate is $x_0 = 0$. Compute numerically (not graphically) the value for x_1, the root estimate after the first iteration of the algorithm is completed.

$x_1 =$

(c) Write down the value of the root that the Newton-Raphson algorithm will find, if the initial root estimate is $x_0 = 2$, by reading its approximate value from the graph.

`Ans:`

[1] Write your answer with only two significant figures, e.g. 2.3, when reading values from the graph.

2. Let x_1, x_2, x_3, and x_4 be the unknown test scores of the four students Erin, Tina, Jack and Ben respectively and denote the vector x as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{where} \quad \begin{array}{l} x_1 \text{ represents Erin's Score} \\ x_2 \text{ represents Tina's Score} \\ x_3 \text{ represents Jack's Score} \\ x_4 \text{ represents Ben's Score}. \end{array}$$

Suppose that you are asked to determine the values of the elements of x given the following information:

- Erin scored 30 points higher than Jack.
- Tina's score is equal to 18 plus the average of Jack and Ben's scores.
- Jack scored 23 points less than the average of Erin and Ben's scores.
- The average of the four scores is 78.

(a) Determine the matrix A and vector b so that x can be determined as the solution of $Ax = b$:

$$A = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix} \qquad b = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}$$

(b) Given that `rank(A) = 4`, circle the correct statement below:

An exact solution of $Ax = b$ `exists` `does not exist`

3. Fig. 1 plots the monthly temperatures of the city of Tucson for two years. The plot was generated using the 24×1 (column) arrays shown below

```
Xvec = (1:24 )' ;

Yvec = [18.9 ; 21.1 ; 23.3 ; 27.8 ; 32.2 ; 37.2 ; 36.1 ; 34.4 ; 29.4 ;23.3 ;18.9 ; ...

18.5  ; 19.7 ; 24.6 ; 27 ; 33.5 ; 36.8 ; 37.6 ; 36.5 ; 33.8 ; 29 ; 23.6 ; 20 ];
```

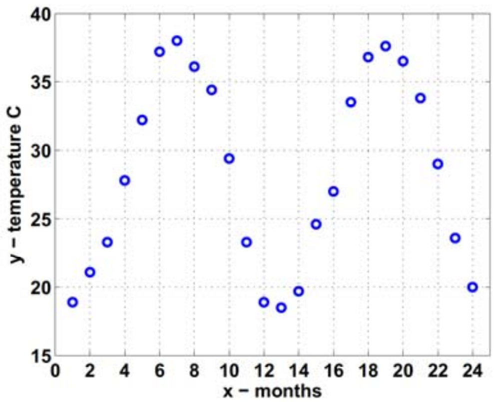


Figure 1: Tucson Temperatures for two years

Suppose we want to fit the data to the following equation (with **3 basis functions**) using least squares regression.

$$\hat{y}(x) = p_1\, f_1(x) + p_2\, f_3(x) + p_3\, f_3(x)\,. \tag{1}$$

It is believed that the general trend of the data, apparent from Fig. 1 is genuine. Based on this belief, which of the proposed basis functions, given as anonymous functions in the 1×6 cell array BF (shown below) would you consider necessary in order to obtain a good fit?

```
BF = [ { @(x) ones(size(x)) }  , { @(x) x } , { @(x) exp(x) } , . . .

{ @(x) x .* x }  , { @(x) cos(pi/6*x) } , { @(x) sin(pi/6*x) } ];
```

(a) Select the three function handles that you would use to perform the regression by specifying the indexes below:

ANS: BF{ } BF{ } BF{ }

(Problem continues on the next page)

(b) Complete the missing lines of code (shown as dashed lines) of the script shown below, which generates Fig. 2 shown below.

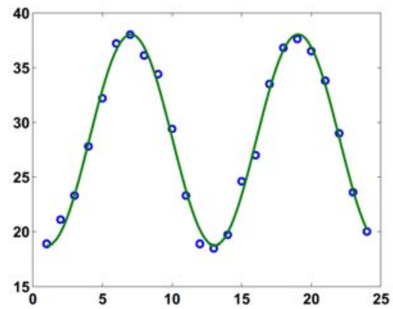


Figure 2: Regression for Tucson Temperatures for two years

- Assume that arrays `Xvec` and `Yvec` and the cell array `BF` in the previous page have been created.
- Using the three function handles you have chosen:
  - Determine the optimal coefficient vector p as the solution to the least squares problem
  - Generate the function handle `yh` to the function $\hat{y}(x)$ in Eq. (1).
- Plot the data points and the regression function evaluated at the values in `Xvec` on the same figure.

begin code

```
% 1)    Create the A matrix to perform least squares regression

A =
    -----------------------------------------------------


% 2)    Obtain the optimal coefficient vector p

p =
    -----------------------------------------------------


% 3)    Create the function handle

yh =
    -----------------------------------------------------


% Create plot

plot( x ,y, 'o', __________ , __________ );
grid on
```

end code

4. The **recursive** function `Bin2dec`, has the following syntax:

```
y = Bin2dec(BA)
```

- `BA` is a $1 \times n$ array of binary digits (i.e. 0's and 1's) e.g. `[1 0 1 0]`.
- `y` is the decimal value of the number represented by the array of binary digits.

Examples:

- `Bin2dec([1 0])` returns 2.
- `Bin2dec([1 1])` returns 3.
- `Bin2dec([0 1 1])` returns 3.
- `Bin2dec([1 0 1 0])` returns 10.

Complete the 2 missing lines of code below:
Non-recursive solutions will not receive credit!

```
begin code
function y = Bin2dec(BA)
n = length(BA);

if n == 1


   y =
        ----------------------------------------------------

else


   y =
        ----------------------------------------------------

end
end code
```

5. Consider the function $f(x) = \frac{1}{8}x^4 - \frac{7}{12}x^3 - \frac{1}{8}x^2 + \frac{31}{12}x$. For your convenience, the table below lists some values.

x	0	1	2	3	4
$f(x)$	0	2	2	1	3

In this problem, we will consider two different approaches to compute

$$\int_0^4 f(x)dx\,.$$

The exact value of the integral is $6\frac{4}{15} \approx 6.2667$.

(a) Use Simpson's rule to estimate the value of the integral [2].

`Answer:`

(b) Use the composite trapezoid rule with 4 subdivisions of $[0, 4]$ to estimate the value of the integral.

`Answer:`

[2]Simpson's rule uses only 3 function evaluations (i.e. no subdivisions of $[0, 4]$)

6. (a) Given 4 data points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) satisfying $x_1 < x_2 < x_3 < x_4$, there is a unique cubic polynomial $c(x)$ that passes through all four points [3], i.e.

$$c(x_i) = y_i \text{ for } i = 1, \ldots, 4.$$

True or False: The cubic-spline interpolation for these same four data points, which also passes through all data points, has the same values as the polynomial $c(x)$ for all values of x between x_1 and x_4.

Write "True" or "False": ____________________

(b) A cubic spline is fit through **6 points**, $(x_1, y_1), (x_2, y_2), \ldots, (x_6, y_6)$ satisfying $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$. This yields 5 cubic polynomials:

- f_1 is valid in the interval $[x_1\ x_2]$,
- f_2 is valid in the interval $[x_2\ x_3]$, and so on, until
- f_5 is valid in the interval $[x_5\ x_6]$.

i. **True or False**: The coefficients of the polynomial f_2 depend on the value of y_3.

Write "True" or "False": ____________________

ii. **True or False**: The coefficients of the polynomial f_1 depend on the value of y_6.

Write "True" or "False": ____________________

iii. A new cubic spline is created, **using only the first 5 data points**, $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$. This yields 4 cubic polynomials:

- g_1 is valid in the interval $[x_1\ x_2]$,
- g_2 is valid in the interval $[x_2\ x_3]$, and so on, until
- g_4 is valid in the interval $[x_4\ x_5]$.

True or False: Since $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are unchanged, it must be that the polynomial f_1 equals the polynomial g_1.

Write "True" or "False": ____________________

(Continues on the next page)

[3] This fact was proven in class.

(c) Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \cdots < x_n$.

Define a **quadratic-spline** interpolation as follows:

- quadratic polynomial f_1 is valid in the interval $[x_1\ x_2]$,
- quadratic polynomial f_2 is valid in the interval $[x_2\ x_3]$ and so on.

The polynomials are required to satisfy the following *constraints*:

- the value of $f_1(x_1) = y_1$
- the value of $f_{n-1}(x_n) = y_n$
- at the intermediate points, the polynomials must pass through the data points, and the slopes of joining quadratics must be equal, i.e.

$$f_{i-1}(x_i) = f_i(x_i) = y_i \qquad f'_{i-1}(x_i) = f'_i(x_i) \qquad \text{for} \quad 2 \leq i \leq n-1\,.$$

Circle the right answer: There are:

- too few
- too many
- the right number

of *constraints* to uniquely determine **all** the coefficients of the interpolating quadratic polynomials.

Explain your reasoning below.

7. Find the error in the following proof that *"all horses are the same color"* [4].

CLAIM: In any set of h horses, all horses are the same color.

PROOF: By induction on h:

Base Case: For $h = 1$. In any set containing just one horse, all horses are clearly the same color.

Induction step: For $k \geq 1$ assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We will now show that all the horses in this set are the same color:

- Remove one horse from this set to obtain the set H_A with just k horses. By the induction hypothesis, all the horses in H_A are the same color.
- Now return the removed horse and remove a different one to obtain a new set H_B with just k horses. By the same argument, all the horses in H_B are the same color.
- Since H_A and H_B have some overlapping horses, it must be that all the horses in H must be the same color, and the proof is complete.

1. Carefully follow the induction steps of the proof when going from two horses to three horses and indicate if there is a step in the proof which is invalid (i.e. start by assuming that, in any set of two horses, all horses are the same color):

   `Answer:`

2. Carefully follow the induction steps of the proof when going from one horse to two horses and indicate if there is a step in the proof which is invalid (i.e. start from the obvious fact that, in any set of one horse, all horses are the same color):

   `Answer:`

[4] From Sipser, *Theory of Computation*, 1997.