1.

Use the Gram-Schmidt process to find an orthonormal basis for $W = Span\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$, where:

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$V_1 = U_1 \qquad \|V_1\| = \sqrt{1^2+0^2+2^2+(-1)^2} = \sqrt{6}$

$$V_2 = U_2 - \frac{V_1 \cdot U_2}{V_1 \cdot V_1} V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{[1\ 0\ 2\ -1]\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}}{[1\ 0\ 2\ -1]\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{\overset{\cancel{4}\ 2}{\cancel{1+0+2+1}}}{\underset{\cancel{6}\ 3}{\cancel{1+0+4+1}}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2/3 \\ 1-0 \\ 1-4/3 \\ -1+2/3 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1/3 \\ 1 \\ -1/3 \\ -1/3 \end{bmatrix} \qquad \|V_2\| = \sqrt{(1/3)^2+1^2+(-1/3)^2+(-1/3)^2} = \sqrt{4/3}$$

$$V_3 = U_3 - \frac{U_3 \cdot V_2}{V_2 \cdot V_2} V_2 - \frac{U_3 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{[0\ 1\ 1\ 0]\begin{bmatrix} 1/3 \\ 1 \\ -1/3 \\ -1/3 \end{bmatrix}}{[1/3\ 1\ -1/3\ -1/3]\begin{bmatrix} 1/3 \\ 1 \\ -1/3 \\ -1/3 \end{bmatrix}} \begin{bmatrix} 1/3 \\ 1 \\ -1/3 \\ -1/3 \end{bmatrix} - \frac{[0\ 1\ 1\ 0]\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}}{[1\ 0\ 2\ -1]\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\overset{2/\cancel{3}}{\cancel{1-1/3}}}{\underset{\cancel{4}/\cancel{3}}{\cancel{1/9+1+1/9+1/9}}}\ \frac{1}{2} \begin{bmatrix} 1/3 \\ 1 \\ -1/3 \\ -1/3 \end{bmatrix} - \frac{\overset{1/3}{0+0+2+0}}{1+0+4+1} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - 1/6 - 1/3 \\ 1 - 1/2 - 0 \\ 1 + 1/6 - 2/3 \\ 0 + 1/6 + 1/3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \|V_3\| = \sqrt{(-1/2)^2+(1/2)^2+(1/2)^2+(1/2)^2} = 1$$

basis: $\begin{bmatrix} 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \frac{1/3}{\sqrt{4/3}} \\ 1/\sqrt{4/3} \\ \frac{-1/3}{\sqrt{4/3}} \\ \frac{-1/3}{\sqrt{4/3}} \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

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2.

Suppose A is a 3×3 matrix with real entries. Are the following statements true or false? Justify your answer.

2a.

If A has 3 linearly independent eigenvectors then it is diagonalizable.

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True, they form the columns of P which will be invertible b/c they are linearly independent.

2.b

If A is diagonalizable it must have 3 distinct eigenvalues.

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False, A can have multiple eigenvectors for one eigenvalue. It is only necessary to have 3 linearly independent eigenvectors - and they don't need to come from distinct λs. example?

2.d

If $A = PDP^{-1}$ then A has the same eigenvalues as D. D is the diagonal matrix

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True b/c D is formed by placing the eigenvalues of A along the diagonal. Also, if a matrix is triangular, its eigenvalues are on the diagonal, so D's eigenvalues are along it's diagonal & are defined as A's eigenvalues.

2.e

If A is diagonalizable then A is invertible.

False For A to be invertible: must have 3 lin ind cols

0 cannot be an eigenvalue

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For A to be diagonalizable must have 3 lin ind eigenvectors

If 0 is an eigenvalue of a diagonalizable matrix, A is not invertible

example?

3.

Consider $M_{2\times 2}$ the vector space of 2×2 matrices (with standard addition and multiplication by scalars). Let

$$F = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}$$

Consider the map $T : M_{2\times 2} \to M_{2\times 2}$ such that for each 2×2 matrix A, $T(A) = AF$.

a) Show that T is a linear transformation.

b) Find the matrix associated to T with respect to the basis $\{b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, b_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

a) $T(A) = AF$

$T(A+B) = (A+B)F = AF + BF = T(A) + T(B)$ ✓

$T(cA) = (cA)F = c(AF) = cT(A)$ ✓

b) $T(b_1)$ $\quad T(b_2)$ $\quad T(b_3)$ $\quad T(b_4)$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & -1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix}$

$[\,T(b_1)\ T(b_2)\ T(b_3)\ T(b_4)\,]$

$$= \begin{bmatrix} -2 & -1 & 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 4 & 3 \end{bmatrix}$$

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4.

Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line that best fits the given data points:

(1,0) , (2,2), (3,1)

$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad y = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}$

$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$X^T y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$(X^T X) B = X^T y$

$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 3 & 6 & 3 \\ 6 & 14 & 7 \end{bmatrix} \begin{matrix} \div 3 \\ -2 \text{ row } 1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} - \text{row } 2 \\ \div 2 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{bmatrix}$

$B_0 = 0 \quad B_1 = 1/2$

$y = 0 + 1/2 x$