Math 54, Spring 2012, F.Rezakhanlou

Each question should be answered directly. Use the back of these sheets if necessary. Justify your assertions; include detailed explanation, and show your work. No aid (including calculators) are allowed.

Your Name: KEY

Your GSI's Name: Jason Ferguson

Your Section: 101 and 104

- **1.** (a)(*20 pts*) Find a matrix P that orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (-\lambda)^3 + 1\cdot1\cdot1 + 1\cdot1\cdot1 - (-\lambda) - (-\lambda) - (-\lambda) = -\lambda^3 + 3\lambda + 2$

Try $\lambda = -1$. $1 - 3 + 2 = 0$. Polynomial division.

$$\begin{array}{r} -\lambda^2 + \lambda + 2 \\ \lambda + 1 \overline{)\, -\lambda^3 + 0\lambda^2 + 3\lambda + 2} \\ +\lambda^3 + \lambda^2 \quad\quad\quad \\ \lambda^2 + 3\lambda \quad \\ -\lambda^2 - \lambda \quad \\ \hline 2\lambda + 2 \end{array}$$

$-\lambda^3 + 3\lambda + 2 = -(\lambda+1)(\lambda^2 - \lambda - 2) = -(\lambda+1)(\lambda+1)(\lambda-2)$.

$Nul(A-(-1)I) \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = -x_2 - x_3 \quad \vec{x} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{u_1} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{u_2}$

Gram-Schmidt: $v_1 = u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. $v_2 = u_2 - \left(\frac{u_1 \cdot u_2}{v_1 \cdot v_1}\right) v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Normalize: $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

$Nul(A-2I) \rightsquigarrow \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Normalize: $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\boxed{\text{If } P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \text{ then } P^{-1} = P^T \text{ and } P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D}$$

(b) (*10 pts*) Find A^4 and find a matrix B such that $B^3 = A$. (Express your answers in terms of P.)

$A^4 = PD^4P^{-1} = \boxed{P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}P^{-1}}$

If $\boxed{B = P\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt[3]{2} \end{bmatrix}P^{-1}}$, then $B^3 = P\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}P^{-1} = A$.

- **2.** (15 points) Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 5 & 10 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 0 & 1 & -5 & -10 \end{bmatrix}.$$

What is the rank of A?

$$A \sim \underset{R_2 \leftrightarrow R_5}{} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 0 & -1 & 5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Col A is Cols 1, 2, 4 of A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 10 \\ 18 \\ 8 \\ -10 \end{bmatrix} \right\}$

rank A = dim Col A = size of basis of Col A = $\boxed{3}$

• **3.** (10 points) Given a nonzero $n \times n$ matrix A, define $<\mathbf{a}, \mathbf{b}> = (A\mathbf{a}) \cdot (A\mathbf{b})$. Under what conditions on A, $<\mathbf{a}, \mathbf{b}>$ defines an inner product for $\mathbb{R}^n$? Explain your answer.

It's an inner product if and only if A is invertible.

$\Rightarrow$ Suppose A is not invertible. Then $A\vec{x} = \vec{0}$ has a nonzero solution $\vec{x}$. Then $\langle \vec{x}, \vec{x} \rangle = (A\vec{x}) \cdot (A\vec{x}) = \vec{0} \cdot \vec{0} = 0$, even though $\vec{x} \neq \vec{0}$. So not an inner product.

$\Leftarrow$ Suppose A is invertible. Then (since dot product is an inner product)

$\langle \vec{x}, \vec{y} \rangle = (A\vec{x}) \cdot (A\vec{y}) = (A\vec{y}) \cdot (A\vec{x}) = \langle \vec{y}, \vec{x} \rangle$.

$\langle \vec{x} + \vec{y}, \vec{z} \rangle = (A(\vec{x} + \vec{y})) \cdot (A\vec{z}) = (A\vec{x} + A\vec{y}) \cdot (A\vec{z}) = (A\vec{x}) \cdot (A\vec{z}) + (A\vec{y}) \cdot (A\vec{z}) = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

$\langle c\vec{x}, \vec{y} \rangle = (A(c\vec{x})) \cdot (A\vec{y}) = (c(A\vec{x})) \cdot (A\vec{y}) = c((A\vec{x}) \cdot (A\vec{y})) = c \langle \vec{x}, \vec{y} \rangle$

$\langle \vec{x}, \vec{x} \rangle = (A\vec{x}) \cdot (A\vec{x}) \geq 0$.

If $\langle \vec{x}, \vec{x} \rangle = 0$, then $(A\vec{x}) \cdot (A\vec{x}) = 0$, so $A\vec{x} = \vec{0}$; since A is invertible we have $\vec{x} = \vec{0}$.

(b) (5 points) Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{b} \neq 0$. Define $T(\mathbf{v}) = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{b} \cdot \mathbf{a}} \mathbf{b}$. Find the dimension of the null space and the range of T.

Claim: Nul(T) is exactly equal to Span$\{\vec{b}\}$.

Suppose $\vec{v}$ is in Nul T. Then $\vec{v} - \left(\frac{\vec{v} \cdot \vec{a}}{\vec{b} \cdot \vec{a}} \vec{b}\right) = 0$, so $\vec{v} = \frac{\vec{v} \cdot \vec{a}}{\vec{b} \cdot \vec{a}} \vec{b}$,

so $\vec{v}$ is a constant multiple of $\vec{b}$. Conversely, for any $c \in \mathbb{R}$,

$T(c\vec{b}) = c\vec{b} - \left(\frac{(c\vec{b}) \cdot \vec{a}}{\vec{b} \cdot \vec{a}}\right)\vec{b} = c\vec{b} - \left(\frac{c(\cancel{\vec{b} \cdot \vec{a}})}{\cancel{\vec{b} \cdot \vec{a}}}\right)\vec{b} = c\vec{b} - c\vec{b} = \vec{0}$.

So Nul T is Span$\{\vec{b}\}$. Since $\vec{a} \cdot \vec{b} \neq 0$, $\vec{b} \neq \vec{0}$, so Span$\{\vec{b}\}$ = Nul T has dimension $\boxed{1}$.

T is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. By the Rank-Nullity Thm,

dim Nul T + dim Ran $T = n$,

so dim Ran $T = \boxed{n - 1}$

- **4. (True - False)** *(20 points)*
For each of the questions below, indicate if the statement is **true** or **false**. If true, **justify** (give a brief explanation or quote a relevant theorem from the course), and if false, give a counter-example or explain.

(a) If $AA^T = A^TA$, then $\|Ax\| = \|A^Tx\|$. Use the fact that $\vec{x}\bullet\vec{y} = \vec{x}^T\vec{y}$.

$\boxed{\text{True}}$ $\|A\vec{x}\| = \sqrt{(A\vec{x})\bullet(A\vec{x})} = \sqrt{(A\vec{x})^T(A\vec{x})} = \sqrt{\vec{x}^TA^TA\vec{x}}$

$\overset{\text{given}}{=} \sqrt{\vec{x}^TAA^T\vec{x}} = \sqrt{(A^T\vec{x})^T(A^T\vec{x})} = \sqrt{(A^T\vec{x})\bullet(A^T\vec{x})}$

$= \|A^T\vec{x}\|$

(b) For all a, b, c, θ

$$(a\cos\theta + b\sin\theta + c)^2 \le 2(a^2 + b^2 + c^2).$$

$\boxed{\text{True}}$ Use Cauchy-Schwarz on $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix}$.

$|\langle\vec{u},\vec{v}\rangle| = |a\cos\theta + b\sin\theta + c|$. $\|\vec{u}\| = \sqrt{a^2+b^2+c^2}$.

$\|\vec{v}\| = \sqrt{\cos^2\theta + \sin^2\theta + 1} = \sqrt{2}$.

Square both sides of $|\langle\vec{u},\vec{v}\rangle| \le \|\vec{u}\|\|\vec{v}\|$ to get $(a\cos\theta + b\sin\theta + c)^2 \le 2(a^2+b^2+c^2)$

(c) If A and B are orthogonal matrices of the same size, then AB is an orthogonal matrix. A, B are invertible, so AB is too.

$\boxed{\text{True}}$ $(AB)^{-1} = B^{-1}A^{-1} \overset{A \text{ and } B \text{ are orthogonal}}{=} B^TA^T = (AB)^T$,

so AB is orthogonal

(d) If λ_0 is an eigenvalue of a matrix A, then the multiplicity of λ_0 as a root of the characteristic polynomial, is the same as the dimension of the eigenspace corresponding to λ_0.

$\boxed{\text{False}}$ This holds if and only if A is diagonalizable.

For a specific counterexample, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Then $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$, so 0 is a double root of the char. poly. of A. But

$\mathrm{Nul}(A - 0I) \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow x_2 = 0$, so $\mathrm{Nul}(A-0I)$ is 1-dimensional, with basis $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.