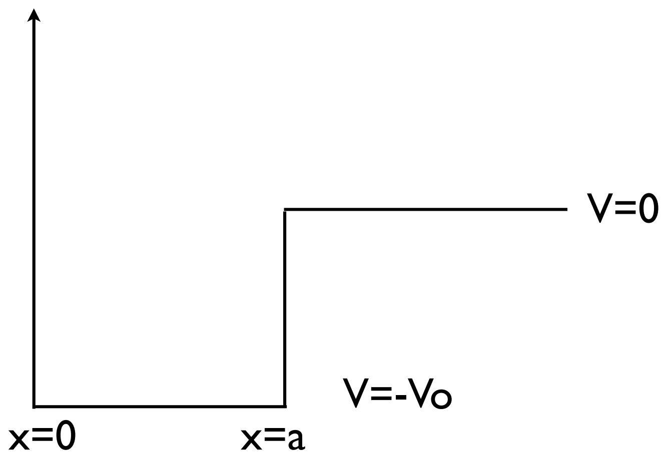
Physics 137A: Second Midterm
Closed Book and Closed Notes: 50 Minutes

1) (20 pts) Review problem: A particle of mass m is confined in the half-infinite, half-finite square well of depth of $Vo = |Vo|$ and width a: $V(x) = \begin{cases} \infty & x < 0 \\ -Vo & 0 < x < a \\ 0 & a < x \end{cases}$



a) (4 pts) Assuming a bound state (E <0), write down appropriate wave functions for the interior ($0 < x < a$) and exterior ($a < x$) regions, taking into account the behavior of the wave function at $x = 0$ and at $x = \infty$. Please denote the wave number for the region $0 < x < a$ and $a < x$ by k and κ, respectively, defining these wave numbers in terms of E, m, $\hbar$, and Vo.

Since the wave function must vanish at the origin and at infinity, the only possibilities are

$$\phi(x) = \begin{cases} A \sin kx, & 0 < x < a \\ Be^{-\kappa x} & x > a \end{cases}$$

where $\kappa = \sqrt{-2mE}/\hbar$ and $k = \sqrt{2m(E + V_0)}/\hbar$.

b) (4 pts) By matching the interior and exterior wave functions and their derivatives at the boundary $x = a$, determine the wave function up to one overall normalization constant and determine the eigenvalue condition.

$$A\sin(ka) = B\exp(-\kappa a) \quad \rightarrow \quad B = A\sin(ka)\exp(\kappa a)$$
$$Ak\cos(ka) = -B\kappa\exp(-\kappa a) = -A\kappa\sin(ka) \quad \rightarrow \quad -k\cot(ka) = \kappa$$

so the wave function :

$$\phi(x) = \begin{cases} A\sin kx, & 0 < x < a \\ A\sin ka\ e^{\kappa(a-x)} & x > a \end{cases}$$

c) (4 pts) By solving the eigenvalue equation for the case where there is only one bound state – which you should place at zero binding energy – determine the condition on the potential parameters (Vo and a) that will guarantee that at least one bound state exists.

$$\kappa \rightarrow 0 \quad k \rightarrow \sqrt{2mV_0}/\hbar \quad \text{so } \cot\left(\frac{a\sqrt{2mV_0}}{\hbar}\right) = 0$$
$$\Rightarrow \sqrt{2mV_0}\frac{a}{\hbar} = \frac{\pi}{2} \text{ for exactly one zero} - \text{energy bound state}$$
$$\text{so } \Rightarrow a^2V_0 > \frac{\pi^2\hbar^2}{8m} \text{ for bound states to exist}$$

d) (4 pts) What is the relationship of this problem to the fully finite well problem of depth Vo and width $2a$, centered on the origin (no calculations necessary here)?

The solutions of this problem are equivalent to the odd solutions of the square well problem: the eigenvalues are the same, and if the solution is merely extended as an odd function to negative $-x$, the wave functions would be the same.

e) (no calculations needed here, either) (4 pts) If we had kept the infinite potential for $x < 0$, but used the potential $V(x) = \frac{1}{2}m\omega^2x^2$ for $x > 0$, what would be the resulting spectrum of allowed eigenvalues E_n?

As in 4d), the solutions would satisfy the harmonic oscillator potential Hamiltonian for $x > 0$ and would vanish at $x = 0$. Thus the solutions would correspond to the odd solutions of the harmonic oscillator problem. The spectrum of all solutions is $E_n = \hbar\omega(n + 1/2)$. This the odd solutions would have the spectrum $E_n = \hbar\omega(n + 1/2)$, n=1,3,5,... or equivalently $E_n = \hbar\omega(2n + 3/2)$, n=0,1,2,...

2. a) (4 pts) $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ are Hermitian operators. Express $(\hat{\mathrm{A}}\hat{\mathrm{B}})^\dagger$ in terms of $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ (that is, $\hat{\mathrm{A}}^\dagger$ and $\hat{\mathrm{B}}^\dagger$ should not appear in your final answer).

$$(\hat{\mathrm{A}}\hat{\mathrm{B}})^\dagger = \hat{\mathrm{B}}^\dagger\hat{\mathrm{A}}^\dagger = \mathrm{B\ A} \tag{1}$$

b) (6 pts) Determine whether the following operator combinations are Hermitian, again assuming $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ are Hermitian. (Please show a proof in each case.)

$\hat{\mathrm{A}}\hat{\mathrm{B}} + \hat{\mathrm{B}}\hat{\mathrm{A}}$: $(\hat{\mathrm{A}}\hat{\mathrm{B}} + \hat{\mathrm{B}}\hat{\mathrm{A}})^\dagger = \hat{\mathrm{B}}^\dagger\hat{\mathrm{A}}^\dagger + \hat{\mathrm{A}}^\dagger\hat{\mathrm{B}}^\dagger = \mathrm{B\ A} + \mathrm{A\ B} = \mathrm{A\ B} + \mathrm{B\ A} \Rightarrow$ Hermitian

$\hat{\mathrm{A}}\hat{\mathrm{B}} - \hat{\mathrm{B}}\hat{\mathrm{A}}$: $(\hat{\mathrm{A}}\hat{\mathrm{B}} - \hat{\mathrm{B}}\hat{\mathrm{A}})^\dagger = \hat{\mathrm{B}}^\dagger\hat{\mathrm{A}}^\dagger - \hat{\mathrm{A}}^\dagger\hat{\mathrm{B}}^\dagger = \mathrm{B\ A} - \mathrm{A\ B} = -(\mathrm{A\ B} - \mathrm{B\ A}) \Rightarrow$ not Hermitian

$i(\hat{\mathrm{A}}\hat{\mathrm{B}} - \hat{\mathrm{B}}\hat{\mathrm{A}})$: $(i\hat{\mathrm{A}}\hat{\mathrm{B}} - i\hat{\mathrm{B}}\hat{\mathrm{A}})^\dagger = -i\hat{\mathrm{B}}^\dagger\hat{\mathrm{A}}^\dagger + i\hat{\mathrm{A}}^\dagger\hat{\mathrm{B}}^\dagger = -i(\mathrm{B\ A} - \mathrm{A\ B}) = i(\mathrm{A\ B} - \mathrm{B\ A}) \Rightarrow$ Hermitian

c) (5 pts) Show that if $\hat{P}$ and $\hat{Q}$ have a common, complete set of eigenvectors $\{f_i\}$, so that $\hat{P}|f_{p_i,q_i}\rangle = p_i|f_{p_i,q_i}\rangle$ and $\hat{Q}|f_{p_i,q_i}\rangle = q_i|f_{p_i,q_i}\rangle$ for all $|f_i\rangle \equiv |f_{p_i,q_i}\rangle$ in the Hilbert space, then $[\hat{P},\hat{Q}] = 0$.

Let $|f_i\rangle \equiv |f_{p_i,q_i}\rangle$ represent any eigenvector in the complete Hilbert space. Then

$$\hat{P}\hat{Q}|f_{p_i,q_i}\rangle = \hat{P}q_i|f_{p_i,q_i}\rangle = q_i\hat{P}|f_{p_i,q_i}\rangle = q_ip_i|f_{p_i,q_i}\rangle \tag{2}$$

Similarly,

$$\hat{Q}\hat{P}|f_{p_i,q_i}\rangle = \hat{Q}p_i|f_{p_i,q_i}\rangle = p_i\hat{Q}|f_{p_i,q_i}\rangle = p_iq_i|f_{p_i,q_i}\rangle \tag{3}$$

Subtracting $\Rightarrow [\hat{P},\hat{Q}]|f_{p_i,q_i}\rangle = 0|f_{p_i,q_i}\rangle$ for all states in the Hilbert space $\Rightarrow [\hat{P},\hat{Q}] = 0$.

3. Consider a two-level system with basis states $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The Hamiltonian matrix in this basis is $H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$; alternatively, $\hat{H} = E(|1\rangle\langle 2| + |2\rangle\langle 1|)$.

a) (5 pts) Find the eigenvalues of H and the corresponding stationary states.

The eigenvalue equation is $\lambda^2 - E^2 = 0 \Rightarrow \lambda = \pm E$. The eigenvectors $|s_\pm\rangle$ can be expanded in the same basis with coefficients to be determined, $|s\pm\rangle = a_\pm|1\rangle + b_\pm|2\rangle$.

$$
\begin{aligned}
H|s_+\rangle &= E(|1\rangle\langle 2| + |2\rangle\langle 1|)(a_+|1\rangle + b_+|2\rangle) = E(b_+|1\rangle + a_+|2\rangle) = E|s_+\rangle = E(a_+|1\rangle + b_+|2\rangle) \\
H|s_-\rangle &= E(|1\rangle\langle 2| + |2\rangle\langle 1|)(a_-|1\rangle + b_-|2\rangle) = E(b_-|1\rangle + a_-|2\rangle) = -E|s_-\rangle = -E(a_-|1\rangle + b_-|2\rangle)
\end{aligned}
$$

Thus we find the normalized eigenvectors (stationary states)

$$
|s_+\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle \qquad |s_-\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle \tag{4}
$$

b) (5 pts) Suppose at time $t = 0$ the system is prepared in the state $|S(t = 0)\rangle = |1\rangle$. Find $|S(t)\rangle$, the solution of the time-dependent Schroedinger equation. Express the result as

$$
|S(t)\rangle = |1\rangle\langle 1|S(t)\rangle + |2\rangle\langle 2|S(t)\rangle
$$

That is, determine $\langle 1|S(t)\rangle$ and $\langle 2|S(t)\rangle$ as simple functions of t, E, and $\hbar$.

From above it is immediate that $|S(0)\rangle >= |1\rangle = \frac{1}{\sqrt{2}}|s_+\rangle + \frac{1}{\sqrt{2}}|s_-\rangle$. Consequently, plugging in the stationary state time dependence,

$$
\begin{aligned}
|S(t)\rangle > &= \frac{1}{\sqrt{2}}|s_+\rangle e^{-iEt/\hbar} + \frac{1}{\sqrt{2}}|s_-\rangle e^{iEt/\hbar} \\
&= \frac{1}{2}(|1\rangle + |2\rangle)e^{-iEt/\hbar} + \frac{1}{2}(|1\rangle - |2\rangle)e^{iEt/\hbar} = \cos(\frac{Et}{\hbar})|1\rangle - i\sin(\frac{Et}{\hbar})|2\rangle
\end{aligned}
$$

c) (5 pts) Using the above result, calculate the probabilities $P_1(t)$ and $P_2(t)$ that a measurement will find that the system $|S(t)\rangle$ in state $|1\rangle$ and state $|2\rangle$, respectively. Hint: check that your calculation satisfies $P_1(t) + P_2(t) = 1$.

$$
P_1(t) = |\langle 1|S(t)\rangle|^2 = \cos^2(\frac{\mathrm{Et}}{\hbar}) \qquad P_2(t) = |\langle 2|S(t)\rangle|^2 = \sin^2(\frac{\mathrm{Et}}{\hbar})
$$

4. a) (5 pts) A nuclear excited state with a most probable energy E decays with a lifetime $\tau_m \sim \Delta t = 10^{-15}$ s. What constraint does the energy uncertainty principle place on ΔE? (Give the answer in eV, using $\hbar = 6.58 \times 10^{-16}$ eV s.)

$$\Delta E \Delta t \geq \frac{\hbar}{2} \Rightarrow \Delta E \geq \frac{6.58 \times 10^{-16} \text{ eVs}}{2 \times 10^{-15}\text{s}} = 0.329 \text{ eV}$$

b) (5 pts) Apply the generalized uncertainty principle, $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$, to the operators $\hat{A} = \hat{x}$ and $\hat{B} = \hat{H} = \hat{p}^2/2m + \hat{V}$ to determine $\sigma_x \sigma_H$. (Hint: your answer should involve $\langle \hat{p} \rangle$.)

$$[\hat{x}, \hat{H}] = x(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)) - (-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x))x = \frac{\hbar^2}{2m}(\frac{d^2}{dx^2}x - x\frac{d^2}{dx^2}) = \frac{\hbar^2}{m}\frac{d}{dx} = \frac{i\hbar}{m}\hat{p}$$

$$\Rightarrow \sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i}\langle[\hat{x}, \hat{H}]\rangle\right)^2 = \left(\frac{\hbar}{2m}\langle\hat{p}\rangle\right)^2$$

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle\hat{p}\rangle|$$

c) (5pts) Return to 3b). Using the operator $\hat{Q} = |1\rangle\langle 1|$ and the expression

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H}, \hat{Q}]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$$

where the expectation value is taken with respect to the state $|S(t)\rangle$, derive an expression for $dP_1(t)/dt$. Is it consistent with what you would calculate directly from your answer in 3c)?

$$\frac{d}{dt}\langle S(t)|1\rangle\langle 1|S(t)\rangle = \frac{d}{dt}|\langle 1|S(t)\rangle|^2 \equiv \frac{dP_1(t)}{dt}$$

$$\begin{aligned}\frac{i}{\hbar}\langle S(t)|[\hat{H}, |1\rangle\langle 1|]|S(t)\rangle &= \frac{i}{\hbar}\left(\langle S(t)|\hat{H}|1\rangle\langle 1|S(t)\rangle - \langle S(t)|1\rangle\langle 1|H|S(t)\rangle\right) \\ &= \frac{iE}{\hbar}\left(\langle S(t)|2\rangle\langle 1|S(t)\rangle - \langle S(t)|1\rangle\langle 2|S(t)\rangle\right)\end{aligned}$$

Consequently,

$$\frac{dP_1(t)}{dt} = \frac{iE}{\hbar}\left(2i\sin(\frac{Et}{\hbar})\cos(\frac{Et}{\hbar})\right) = -\frac{2E}{\hbar}\sin\left(\frac{Et}{\hbar}\right)\cos\left(\frac{Et}{\hbar}\right) = -\frac{E}{\hbar}\sin\left(\frac{2Et}{\hbar}\right) \tag{5}$$

We have used $\hat{H} = E(|1\rangle\langle 2| + |2\rangle\langle 1|)$ in the above. Indeed, this is the same answer we get by taking the derivative of the answer in 3c).