Stat134, Lec 3: Midterm I Solutions

Problem 1: (a) Let X be the number of people who prefer Candidate A.

$$\mathbb{P}(530 \leq X \leq 570) = \sum_{k=530}^{570} \binom{1000}{k} p^k (1-p)^{1000-k}$$

(b)Using the Normal Approximation Method, $\mu = 1000p$ and $\sigma = \sqrt{1000p(1-p)}$.

$$\mathbb{P}(530 \leq X \leq 570) \approx \Phi(\frac{570 + .5 - 1000p}{\sqrt{1000p(1-p)}}) - \Phi(\frac{530 - .5 - 1000p}{\sqrt{1000p(1-p)}})$$

To find the maximum point, take the derivative of the above expression and let it be 0:

$$\phi(\frac{570 + .5 - 1000p}{\sqrt{1000p(1-p)}})\Big[\frac{141p - 570.5}{2\sqrt{1000}(p(1-p))^{3/2}}\Big] - \phi(\frac{530 - .5 - 1000p}{\sqrt{1000p(1-p)}})\Big[\frac{59p - 529.5}{2\sqrt{1000}(p(1-p))^{3/2}}\Big] = 0$$

$$\Rightarrow \exp\{-\frac{410\sqrt{10}(0.55-p)}{p(1-p)}\} = \frac{59p - 529.5}{141p - 570.5}$$

$$\Rightarrow p \approx 0.5501$$

Although it's computationally hard, the idea is simple.

(c) Since $\Phi(-2,2) \approx 95\%$, the largest "reasonable" p is corresponding to the point such that $\frac{570+.5-1000p}{\sqrt{1000p(1-p)}} = -2$ and the smallest "reasonable" p is corresponding to the point such that $\frac{530-.5-1000p}{\sqrt{1000p(1-p)}} = 2$.

So the range of "reasonable" p is approximately [0.50, 0.60]

Problem 2: (a) $\binom{365}{k}(\frac{1}{365})^k(\frac{364}{365})^{365-k}$

(b) In this case, $n = 365$, $p = 1/365$, so $\mu = np = 1$

(c)(d)(e)

$$\mathbb{P}(\text{Total \# of successes fro } E_1 \text{ and } E_2 \text{ together is } k)$$
$$= \sum_{i=0}^{k} \mathbb{P}(E_1 \text{ has } i \text{ successes})\mathbb{P}(E_2 \text{ has } k-i \text{ successes})$$
$$= \sum_{i=0}^{k} e^{-1}\frac{1}{i!}e^{-1}\frac{1}{(k-i)!} = e^{-2}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}$$
$$= e^{-2}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i} = e^{-2}\frac{1}{k!}2^k \quad \text{(by the Binomial Formula)}$$

which has Poisson distribution with parameter $\lambda = 2$.

$$\mathbb{P}(1) = 2e^{-2},\ \ \mathbb{P}(2) = 2e^{-2},\ \ \mathbb{P}(3) = \frac{4}{3}e^{-2},\ \ \mathbb{P}(4) = \frac{2}{3}e^{-2},\ \ \mathbb{P}(5) = \frac{4}{15}e^{-2}$$

Problem 3:

$$\mathbb{P}(\text{exactly 2 people have the same birthday}) = \binom{n}{2}\frac{365 \cdot 1 \cdot 364 \cdots (364 - (n-2) + 1)}{365^n}$$

Problem 4: (a)

$$\mathbb{P}(+,+,@,@,*) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}$$

$$\mathbb{P}(+,+,+,@,@) = \frac{13 \cdot 12 \cdot \binom{4}{2}\binom{4}{3}}{\binom{52}{5}}$$

$$\mathbb{P}(+,+,+,+,@) = \frac{13 \cdot \binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

(b) $\mathbb{P}(\text{both hands have at least one pair})$
$=1 - \mathbb{P}(\text{1st hand has no pair}) - \mathbb{P}(\text{2nd hand has no pair}) + \mathbb{P}(\text{both hands have no pair})$

$$\mathbb{P}(\text{1st hand has no pair}) = \mathbb{P}(\text{2nd hand has no pair}) = \frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}} \approx 0.5071$$

$\mathbb{P}(\text{both hands have no pair})$

$=\mathbb{P}(\text{1st hand has no pair})\mathbb{P}(\text{2nd hand has no pair} \mid \text{1st hands has no pair})$

$$=\frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}} \frac{\sum_{k=0}^{5}\binom{8}{5-k}\binom{4}{1}^{5-k}\binom{5}{k}\binom{3}{1}^k}{\binom{47}{5}} \approx 0.2588$$

$\Rightarrow \mathbb{P}(\text{both hands have at least one pair}) \approx 0.2446$

Problem 5: By independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
And disjoint means $A \cap B = \emptyset$, so $\mathbb{P}(A \cap B) = 0$.
Thus, $\mathbb{P}(A)\mathbb{P}(B) = 0$, which indicates at least one of $\mathbb{P}(A)$, $\mathbb{P}(B)$ must be 0.