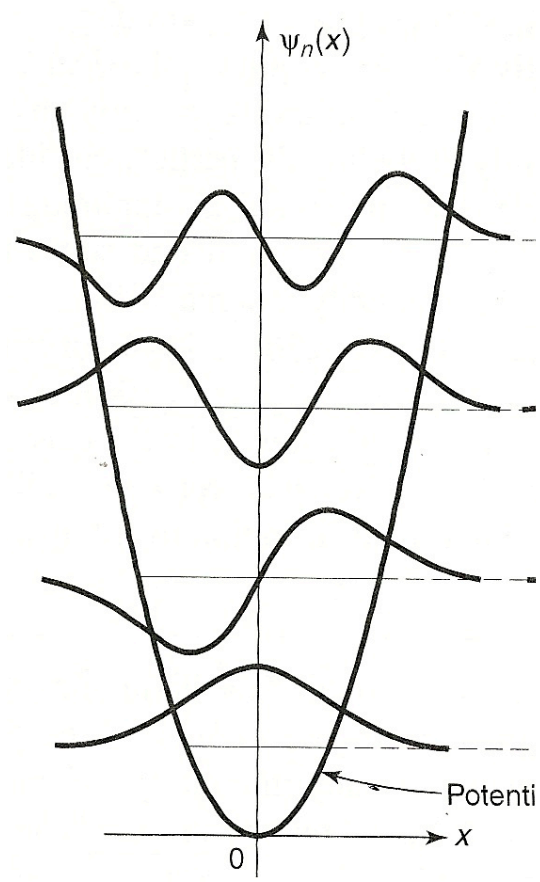
Physics 137A: First Midterm
Closed Book and Closed Notes: 50 Minutes

The first four problems require no or minimal calculations.

1. The y-axis on the harmonic oscillator potential below denotes energy in units of $\hbar\omega$. Draw four straight dashed horizontal lines across the potential at the energies corresponding to the first four eigenstates. Then draw the corresponding $\psi(x)$s for these four states, using the dashed lines as the x-axes around which you display the wave function. (So the positive part of the wave function should be above the axis, the negative part below – don't worry about wave function height or any detail like that. I want to see whether you know what the wave function looks like qualitatively.) (10 pts)

What was desired is a drawing like that given in the book, with energies at $\hbar\omega/2$, $3\hbar\omega/2$, $5\hbar\omega$, and $7\hbar\omega$ and with wave functions having the correct number of nodes, proper positive and negative parts, and some extension into the forbidden region. The book's drawing is:



(a)

2. (8 pts) At time t=0 one is given a wave packet $\phi(x,0)$ for a free particle, with a probability density that is Gaussian and centered at x=0. We showed in class that such a wave packet could be represented by a sum over plane waves, where the expansion coefficients can be determined by doing the Fourier transform.

$$\phi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(p)\; e^{ipx} dp \tag{1}$$

a) At t=0, what is $\langle p \rangle$? Why is this so? (E.g., what does the answer say about the $a(p)$?)

This problem was in problem set 4 (2.22) and was also done in class.

$\langle p \rangle$=0. As the wave packet is symmetric, it will be made up of an equal number of left- and right-going plane waves. This is equivalent to stating that $a(p) = a(-p)$.

b) How does $\langle p^2 \rangle$ evolve with time? How do you know this?

$\langle p^2 \rangle$ stays constant in time. We know for all such packets that $\langle E \rangle$ is constant. But since $E_p = \frac{p^2}{2M}$ for each stationary state making up our packet, $\langle p^2 \rangle$ must also be constant.

c) What will happen to such a wave packet as time goes on? If I had made the original wave packet $\phi(x,0)$ narrower, qualitatively how would this wave packet evolve in time compared to the original, fatter one? (I just want a qualitative answer comparing the two packets.)

We explored this problem in an animation shown in class. The wave packet spreads out in coordinate space, according to the typical momentum scale $\sqrt{\langle p^2 \rangle}$ in the packet: the packet has an equal number of left-going and right-going waves. The narrower the packet, the higher the typical momentum scale, and thus the faster the spreading (see below).

d) Qualitatively, for either wave packet, if I evaluate the uncertainty $\sigma_x \sigma_p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle}$ as a function of time, how will this quantity change? What is the t=0 value (when the packet is a Gaussian)?

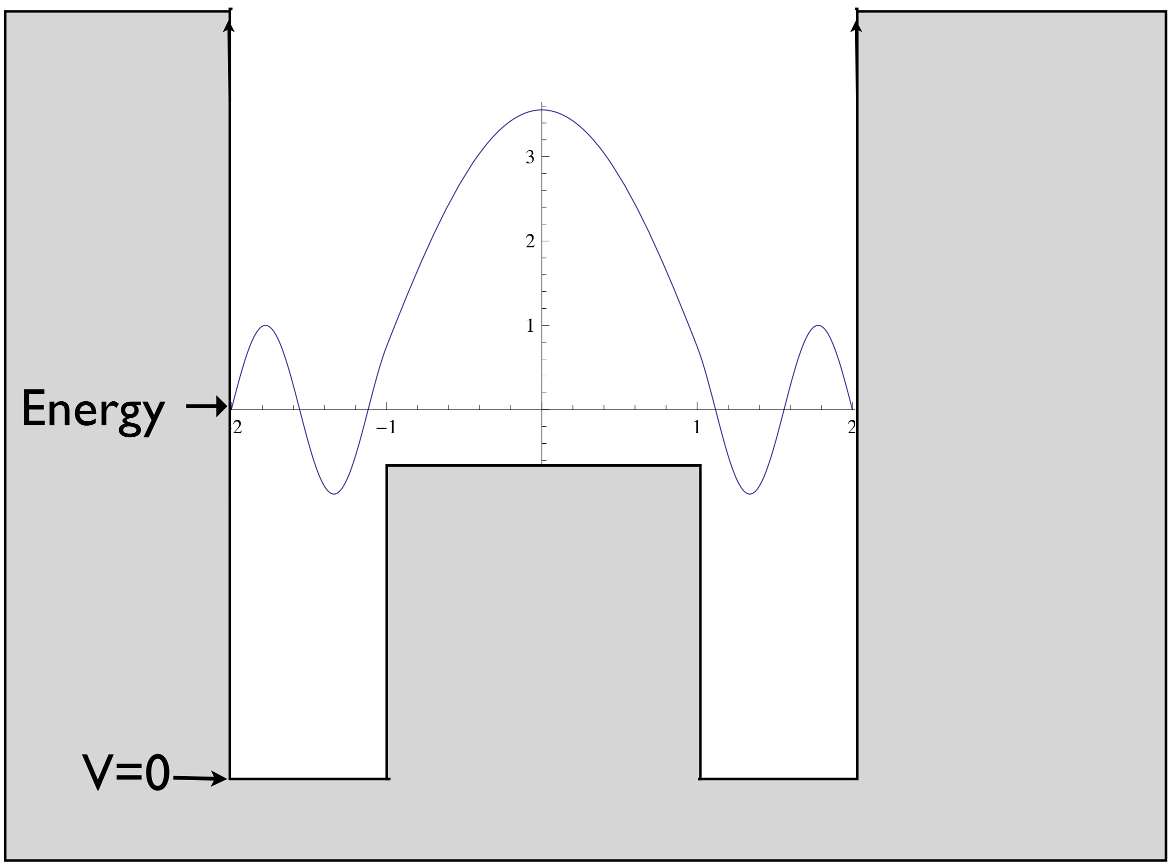
A Gaussian wave packet is a minimum uncertainty wave packet, so the initial value is $\hbar/2$. It is enough to say that the uncertainty will rapidly increase with time because it spreads in coordinate space while $\langle p^2 \rangle$ is constant. If you remember your solution to problem 2.22, even better. The functional dependence in time is

$$\sqrt{\langle p^2 \rangle \langle x^2(t) \rangle} = \hbar/2 \left[1 + \left(\frac{\hbar t}{2\langle x^2(0) \rangle m} \right)^2 \right]^{1/2} \tag{2}$$

where $\langle x^2(0) \rangle$ determines the t=0 wave packet width. Thus as said above, the narrower the initial packet, the faster the spreading.

3. Again NO CALCULATIONS! This is a qualitative problem. The potential below is an infinite square well modified by the repulsive "box" shown. It is observed that the fifth eigenstate has an energy just above the top of the box. The dashed line is the eigenvalue. Make a qualitative drawing of $\psi(x)$ as we have described below: positive parts above the line, negative parts below. I'm interested in questions like the number of nodes (zeros in the wave function), their qualitative locations, etc. (10 pts)

So what I am looking for here is a pattern where the wavelength is shorter in the two deep parts of the well and longer in the shallow part, reflecting the changing wave numbers, as well as a pattern with the appropriate number of nodes. The graph is from an actual solution (which you know how to work out) with $z_0 = \frac{a\sqrt{2mV_0}}{\hbar} = 7$ where each of the deep wells has width a and the middle guy width $2a$.



4. (12 pts) A wave packet $\phi(x, 0)$ can be expanded in terms of the first three harmonic oscillator stationary states ψ_i^{HO} as follows: $\phi(x, 0) = \psi_1^{HO}(x)/\sqrt{2} + \psi_2^{HO}/2 + \psi_3^{HO}/2$.

4.1) In terms of the harmonic oscillator ground state energy E_1, what is $\langle H \rangle$ at t=0?

$$\langle E \rangle = \sum_{n=1}^{\infty} a_n^2 E_n = (1/\sqrt{2})^2 E_1 + (1/2)^2(3E_1) + (1/2)^2(5E_1) = \frac{5}{2}E_1 = \frac{5}{4}\hbar\omega \quad (3)$$

4.2) Qualitatively, how will the expected value of the energy evolve with time?

It will be constant in time.

4.3) Qualitatively, how will $\langle x \rangle$ evolve with time? Explain.

This problem is quite similar to Problem Set 3, problem 2.13. As in that case, because we have both odd and even states, there will be an initial nonzero $\langle x \rangle$ and it will oscillate in time according to the available beat frequencies.

4.4) Can you give earliest time t>0, in units of $\hbar/E_1$, when $\phi(x, t)$ will again be identical to $\phi(x, 0)$?

There are three beat frequencies corresponding to $(E_2 - E_1)/\hbar = 2E_1/\hbar$, $(E_3 - E_2)/\hbar = 2E1/\hbar$, and $(E_3 - E_1)/\hbar = 4E_1/\hbar$. So the last repeats every $\delta\mathrm{t} = (\pi/2)(\hbar/E_1)$ and the first two every $\mathrm{t} = \pi(\hbar/E_1)$. So the wave function returns to its t=0 form after a time π in units of $\hbar/E_1$, or equivalent $2\pi/\omega$ where ω is the oscillator frequency.

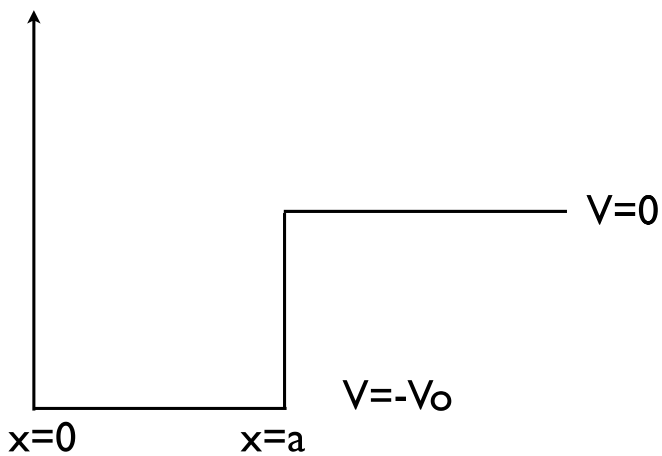
4.5) If the initial wave packet $\phi(x, 0)$ had been given by $\psi_1^{HO}(x)/\sqrt{2} + \psi_3^{HO}(x)/\sqrt{2}$, what would have been the answer to 4.3)? Why?

$\langle x(\mathrm{t}) \rangle = 0$. The wave function is a combination of two symmetric stationary states, so the expectation of x is always 0, as x is an odd function.

4.6) If the initial wave packet $\phi(x, 0)$ had been given by $\psi_1^{HO}(x)/\sqrt{2} + \psi_3^{HO}(x)/\sqrt{2}$, what would have been the answer to 4.4)?

From the above calculation, $\delta\mathrm{t} = (\pi/2)(\hbar/E_1)$ or, equivalently, π/ω.

5) (20 pts) Finally, a problem where you can do algebra! A particle of mass m is confined in the half-infinite, half-finite square well of depth of Vo and width a:



a) Write down an appropriate bound state (E<0) wave function for the interior and exterior regions.

Since the wave function must vanish at the origin and at infinity, the only possibilities are

$$\phi(x) = \begin{cases} A\sin kx, & 0 < x < a \\ Be^{-\kappa x} & x > a \end{cases}$$

where $\kappa = \sqrt{-2mE}/\hbar$ and $k = \sqrt{2m(E+V_0)}/\hbar$.

b) By matching the interior and exterior wave functions and their derivatives at the boundary x=a, determine the wave function up to one overall normalization constant and determine the eigenvalue condition.

$$A\sin(ka) = B\exp(-\kappa a) \quad \rightarrow \quad B = A\sin(ka)\exp(\kappa a)$$
$$Ak\cos(ka) = -B\kappa\exp(-\kappa a) = -A\kappa\sin(ka) \quad \rightarrow \quad -k\cot(ka) = \kappa$$

so the wave function :

$$\phi(x) = \begin{cases} A\sin kx, & 0 < x < a \\ A\sin ka\; e^{\kappa(a-x)} & x > a \end{cases}$$

c) By solving the eigenvalue equation for a zero-energy bound state, determine the condition on the potential parameters (Vo and a) for a bound state to exist.

$$\kappa \to 0 \quad k \to \sqrt{2mV_0}/\hbar \quad \text{so} \;\; \cot\left(\frac{a\sqrt{2mV_0}}{\hbar}\right) = 0$$

$$\Rightarrow \sqrt{2mV_0}\frac{a}{\hbar} = \frac{\pi}{2} \Rightarrow a^2 V_0 = \frac{\pi^2\hbar^2}{8m}$$

d) What is the relationship of this problem to the fully finite well problem of depth Vo and width 2a, centered on the origin (no calculations necessary here)?

The solutions of this problem are equivalent to the odd solutions of the square well problem: the eigenvalues are the same, and if the solution is merely extended as an odd function to negative $-x$, the wave functions would be the same.

**6) (5pts): Only attempt this problem if you have finished the other parts of the exam because this problem is hard and only gives you 5 points. A ping pong ball is dropped from a height equal to ten times its radius onto a second sphere of the same radius. The bouncing of the ping pong ball on this second sphere is perfectly elastic. Also imagine this is done in some ideal environment – no air, no earth rotation, nothing that can spoil the conditions for you. You hit the second sphere dead on, the ping pong recoils up to its original height, then again bounces, and then again, ... What limit does the uncertainty principle place on the number of bounces that might occur?

The problem was intended for anyone who unaccountably finished the rest of the exam with time to spare. The basic idea is that on release, there is an uncertainty in the horizontal position and horizontal momentum (call the horizontal the x direction) that governs the subsequent classical motion of the ping-pong ball. Thus it will hit a point r_1 some distance from dead center. r_1 is governed by the uncertainty principle. Then this off-center strike will be magnified in the next bounce: the ping-pong ball reflects off the fixed ping-pong ball's surface, with incoming and outgoing angles symmetric around the a line perpendicular to the tangent point. Thus two initial calculations are necessary; 1) how big is r_1? 2) given an r_1 what is the subsequent r_2 where the next collision with the fixed ball occurs? These distances all refer to displacement in x, and all bounces occur at small angles (see below). As the ball is dropped from high up (10 times the radius) there is plenty of time between bounces for deviations to grow. We find

$$r_1 = \Delta x + \frac{\Delta p_x}{m}\frac{\tau}{2}$$

$$r_2 = r_1 + \frac{r_1}{r} v_0 \tau \rightarrow r_2 = r_1 + 40 r_1$$

$$\text{as } \tau = 2\sqrt{\frac{20r}{g}} \text{ and } v_0 = \sqrt{20gr}$$

where in the above, τ is the time between bounces and v_0 is the speed at collision, both determined by the height $10r$ (r the ball radius) from which the ping-pong ball is dropped and by g (acceleration constant for gravity). Subsequent bounces just repeat the cycle. Note also that since the horizontal displacement increases by a big factor of 41 in each bounce, all bounces are "small angle" bounces: once the ball gets out to 5 degrees off center, the next rebound will miss. I used small angles to simplify the results above, e.g., always treating τ as if it corresponding to a vertical drop. This is a great approximation. From the above the number of bounces n before the ping-pong ball flies off is thus determined by the condition that the displacement is than the radius, and by the initial r_1 that is determined by the uncertainty principle. These two considerations translate to

$$\text{find maximum n such that } \; (41)^{n-1} r_1 < r$$

$$\text{minimize } r_1 = \Delta x + \frac{\hbar}{2\Delta x m}\frac{\tau}{2} \Rightarrow 0 = 1 - \frac{\hbar}{2(\Delta x_{\text{min}})^2 m}\frac{\tau}{2} \Rightarrow \Delta x_{\text{min}} = \frac{1}{2}\sqrt{\frac{\hbar\tau}{m}}$$

where in the second line we use the uncertainty principle, then look for the minimum in r_1 by solving for the inflection point. Plugging in reasonable numbers (3 grams for the mass, 2 cm for the radius) I get $\Delta x = 6.6 \times 10^{-13}$ cm and $r_1 = 1.3 \times 10^{-14}$ cm. So as in baseball, it is all over except for the shouting

$$(41)^{n-1}(1.3 \times 10^{-14}\text{cm}) < 2 \text{ cm} \Rightarrow$$

$$\text{so the maximum number of bounces is } 9$$

(Note n=1 is the first bounce ($r = r_1$)). So give it a try - see if you can beat Mr. Heisenberg! Anyone getting anywhere close to a reasonable answer for this under test conditions has really done well!