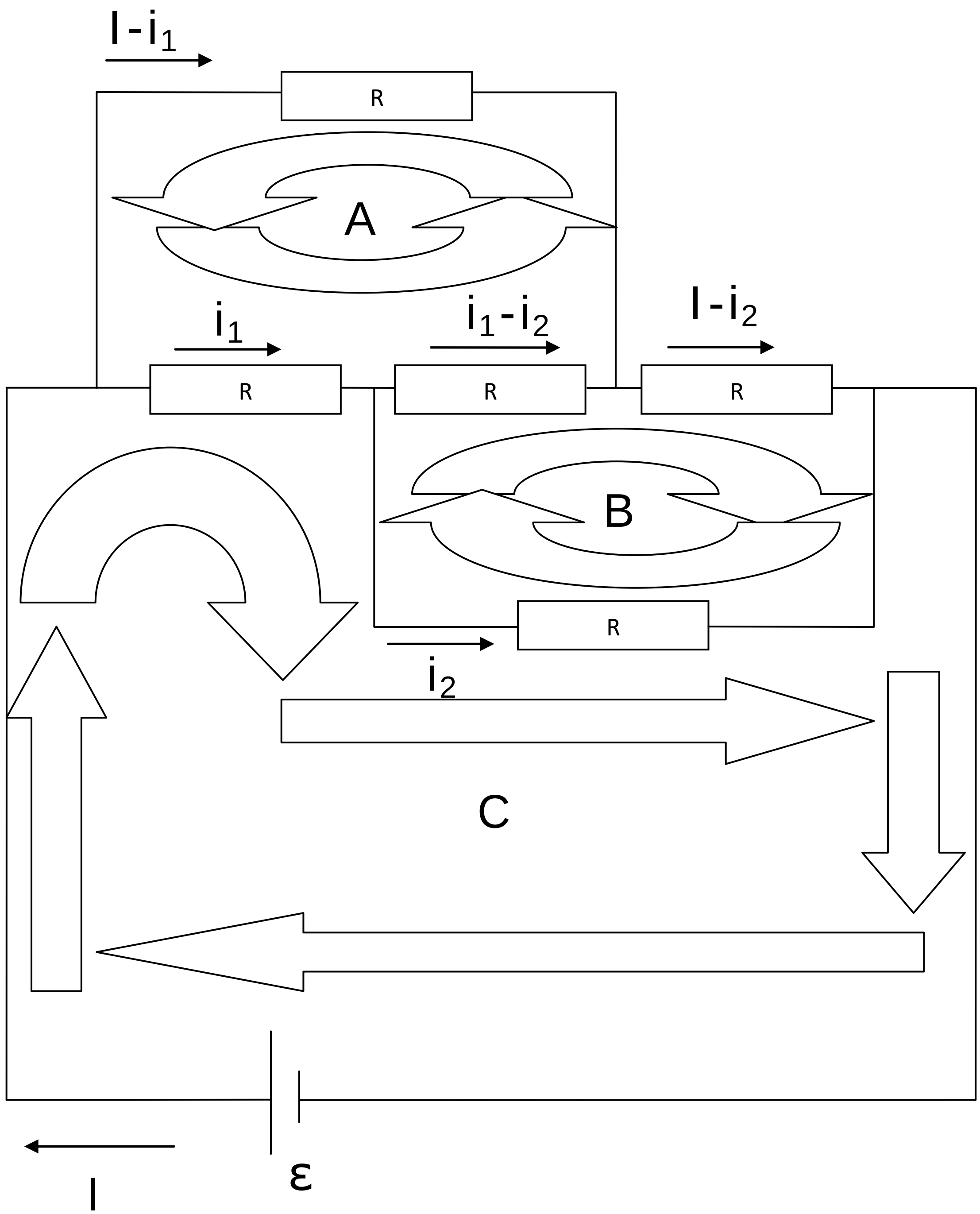
Physics 7B, Lecture 001, Spring 2012 (Xiaosheng Huang)

Mid-term 2 – Problem 2 solution

Method 1:

Using Kirchoff's current law, we can label the current flowing through each resistor. i_1 and i_2 are unknown currents that have been introduced.



Three loops have also been drawn in the above diagram. Using Kirchoff's voltage law, we know that the voltage drop around any loop must sum to 0.

<u>Loop A:</u>

$R*i_1 + R*(i_1-i_2) - R*(I-i_1) = 0$

⇨ $3i_1 - i_2 = I$

<u>Loop B:</u>

$R*(i_1-i_2) + R*(I-i_2) - R*(i_2) = 0$

⇨ $I = 3i_2 - i_1$

If we equate the "I" in both equations:

$3i_1 - i_2 = 3i_2 - i_1$

$i_1 = i_2$

Subbing them back into our "I" equations would tell us that

$i_1 = i_2 = I/2$

<u>Loop C:</u>

$R*i_1 + R*i_2 - \varepsilon = 0$

$\varepsilon = R*(i_1 + i_2) = R*I$ since $i_1 = i_2 = I/2$

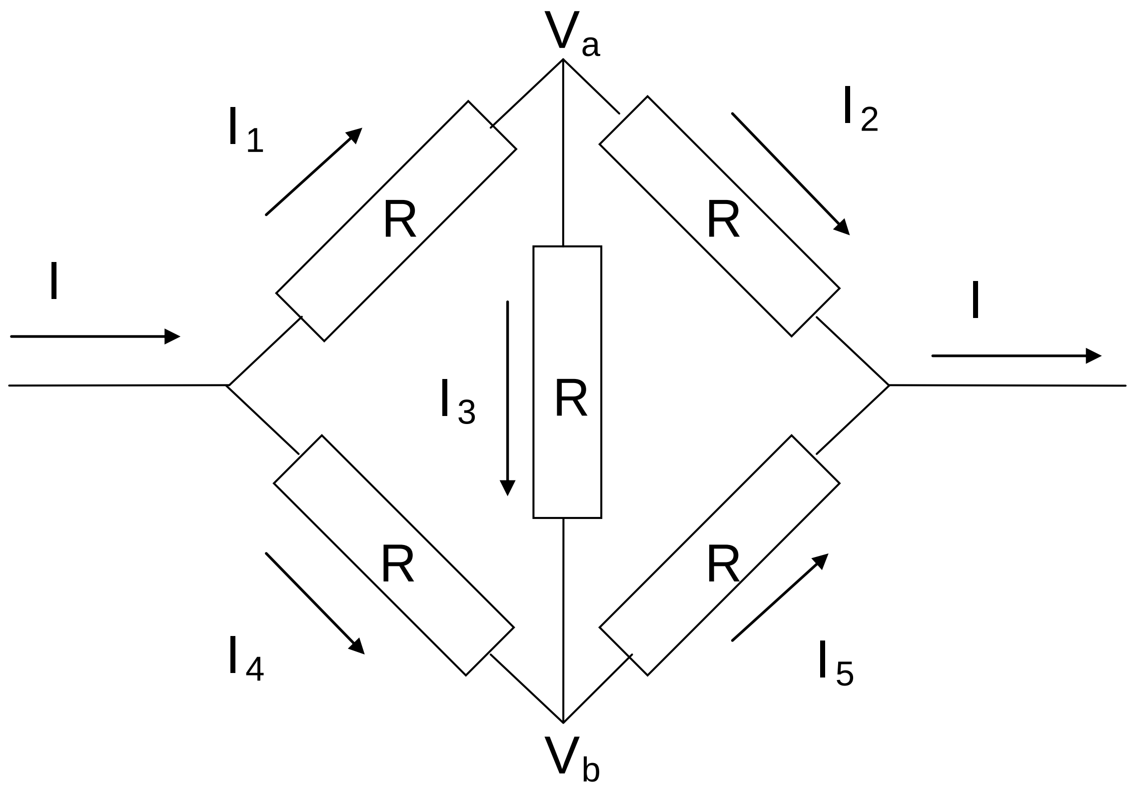
We also know that the EMF is equal to the total potential drop in the circuit

$\varepsilon = V = R_{eff} * I$

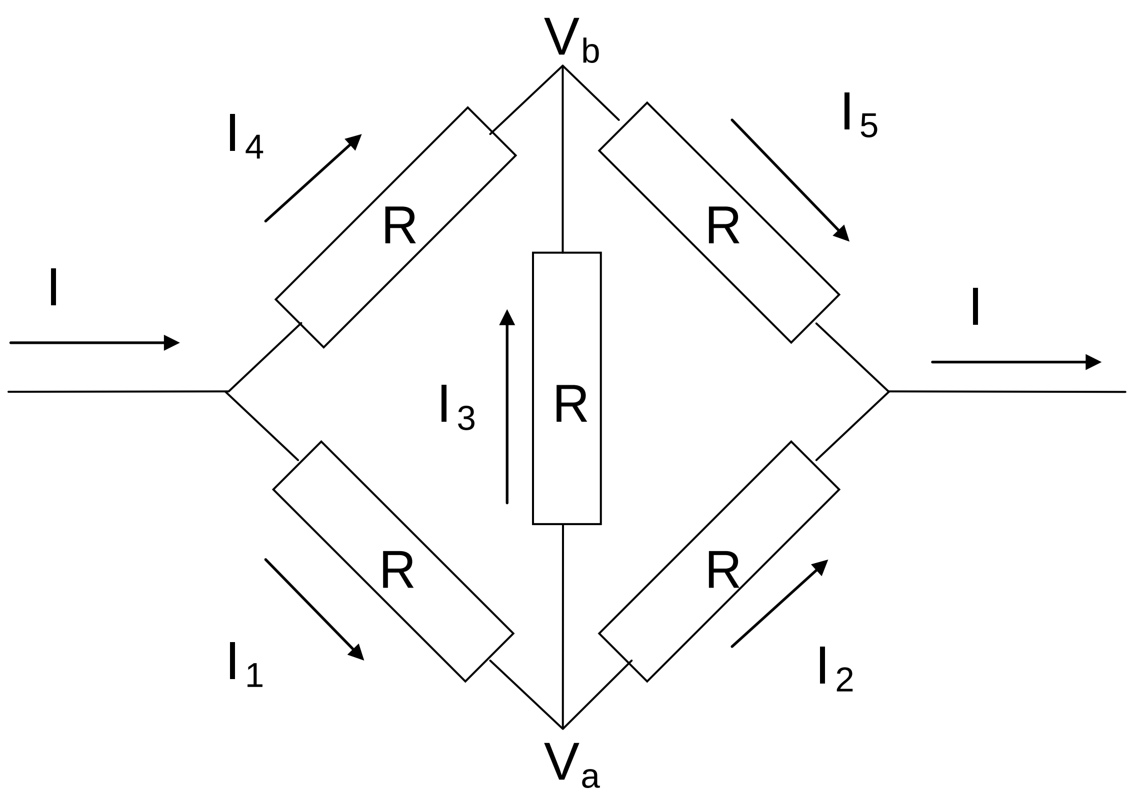
Hence we immediately see that $R_{eff} = R$

Method 2:

By redrawing the circuit, we can easily see that the circuit has a symmetry we can exploit to solve the problem.



Now the clever trick is to flip the diagram upside-down.

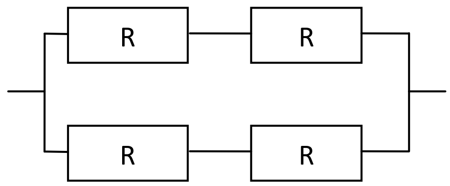


But after the reflection, the assembly of these 5 resistors looks exactly the same as before. Hence they are the same circuit.

We see that the direction of the current through the middle resistor (I_3) is flipped, so the only way to ensure that both circuit are equivalent is for $I_3=0$

We could have also argued that by symmetry $V_a=V_b$. There is no voltage drop across the middle resistor hence no current flows through it.

Since there is no current through the middle resistor, its removal will not affect our circuit.



The effective resistance can be found by summing the resistors in the above diagram, which gives:

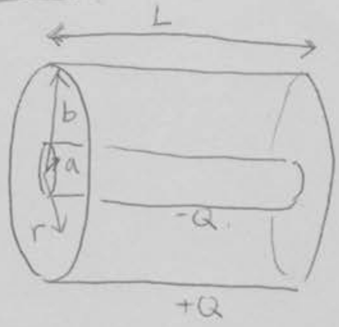
$R_{eff} = R$

<u>An aside:</u>

Symmetry also tells us that $I_1=I_4$ and $I_2=I_5$. Together with Kirchoff's junction rule, we can quickly deduce that $I_1= I_2= I_4= I_5=I/2$ with minimal algebra.

The lesson is that symmetry is a powerful tool and should be exploited whenever possible!

prob 3



(a) (3pt) From either Gauss's law or realizing for this kind of cylindrical symmetric configuration, the $\mathbb{E}$ field is the same as having all the charge as a single line charge density at the center. (1pt)

$$\Rightarrow \mathbb{E}(r) = \frac{-Q/L}{2\pi\epsilon_0 r}\hat{r} = \frac{-Q}{2\pi\epsilon_0 r \cdot L}\hat{r} \quad \text{(2pt)}$$

(b) (4pt). $V(r) = \int_b^r \mathbb{E}(r')\cdot dr' = \int_b^r \frac{+Q}{2\pi\epsilon_0 L}\frac{1}{r'}dr' = \frac{+Q}{2\pi\epsilon_0 L}\log r/b$

(note. this is a negative value when $r<b$) (sign: 2pt)

(c) $C = \frac{Q}{|V_{ab}|} = \frac{Q}{\frac{Q}{2\pi\epsilon_0 L}\log b/a} = \frac{2\pi\epsilon_0 L}{\log b/a}$. (sign 2pt)

(3pt)

(d). (3pt) $U = \frac{1}{2}QV$. w/. Q and V related by $C = \frac{Q}{V}$ or $V = \frac{Q}{C}$. one gets.

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2}\cdot\frac{\log b/a}{2\pi\epsilon_0 L}$$

(1pt partial credit for any of the equivalent form $U = \frac{Q^2}{2C}$ or $U = \frac{CV^2}{2}$)

(-2pt if your energy is negative!).

(e) 6pt

this is the same as two capacitors in parallel.

the right part has capacitance $C_R = \frac{2\pi\epsilon_0(L-x)}{\log b/a}$ (1pt)

the left part (w/. dielectric material) $C_L = \kappa\cdot\frac{2\pi\epsilon_0 x}{\log b/a}$

$$\therefore C_{tot} = C_R + C_L = \frac{2\pi\epsilon_0}{\log b/a}(L - x + \kappa x) \quad \text{(1pt)}$$

(1pt.)

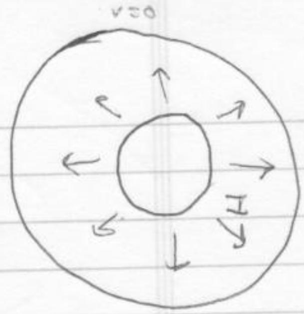
$$\therefore U(x) = \frac{Q^2}{2C(x)} = \frac{Q^2}{2}\cdot\frac{\log(b/a)}{2\pi\epsilon_0(L+(\kappa-1)x)} \quad \text{(3pt)}$$

(f) (3pt) the force acting on the dielectric material is (Not understanding V changes -3pt)

$$F = -\frac{\partial U}{\partial x} = +\frac{Q^2}{4\pi\epsilon_0}\log(b/a)\,\frac{\kappa-1}{(L+(\kappa-1)x)^2} > 0$$

i.e. it's draging the dielectric material to the right! Since by doing so it reduces energy!

4.

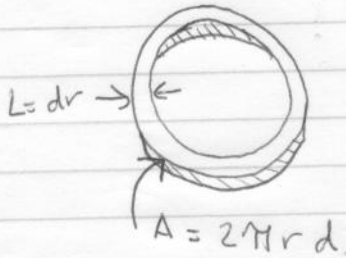


a) $I_{tot} = ?$

$V = IR$; $I = \frac{V}{R}$ ← V_0, ← $R = ?$

Generally, $R = \rho \frac{L}{A}$

Here for a slice of the resistor:



$R = \sum$ slices

$$= \int_{r=a}^{b} \frac{\rho}{2\pi r d}\, dr$$

$$= \frac{\rho}{2\pi d} \int_a^b \frac{1}{r}\, dr$$

$$= \frac{\rho}{2\pi d} \ln\left(\frac{b}{a}\right)$$

$$I = \frac{V_0}{\frac{\rho}{2\pi d} \ln\left(\frac{b}{a}\right)} = \boxed{\frac{2\pi d\, V_0}{\rho \ln\left(\frac{b}{a}\right)}}$$

b) $V(r) = ?$

voltage drop over resistor $V = IR$

$V(r) = I\,R(r)$

voltage drop from r to b!

$$\underbrace{V(r) - V(b)}_{0} = I \underbrace{R_{r\to b}}_{\text{resistance from } r \text{ to } b}$$

$$R_{r\to b} = \int_{r'=r}^{b} \frac{\rho\, dr'}{2\pi r' d} = \frac{\rho}{2\pi d} \int_r^b \frac{1}{r'}\, dr'$$

$$= \frac{\rho}{2\pi d} \ln\left(\frac{b}{r}\right)$$

$$V(r) - 0 = \underbrace{\left(\frac{2\pi d\, V_0}{\rho \ln\left(\frac{b}{a}\right)}\right)}_{I \text{ from pt. a}} \underbrace{\left(\frac{\rho}{2\pi d} \ln\left(\frac{b}{r}\right)\right)}_{R_{r\to b}}$$

$$\boxed{V(r) = V_0 \frac{\ln(b/r)}{\ln(b/a)}} = V_0 - \overbrace{V_0 \frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)}}^{\Delta V_{a\to r}} = V_0 + V_0 \frac{\ln\left(\frac{a}{r}\right)}{\ln\left(\frac{b}{a}\right)}$$

(other versions)

c) same form as 3b with sign difference

Problem 4 Grading Notes

Holistic Scale

Part a	Part b	Description
10	9	student shows mastery of the material and answers all questions with no significant errors
8	7	student shows good understanding, with a few minor errors or calculational mistakes
6	5	student shows reasonable understanding, but with a significant error or omission, or several minor ones
4	3	student shows some working understanding, but with notable errors or several large omissions
2	1	student shows heavily-flawed understanding or omits a significant fraction of the write-up or solution
0	0	student made little or no meaningful attempt to solve the problem

Note on (b) if you tried to integrate the E field:
Gauss's law cannot be applied w/o justifying why this is equivalent to a cylinder (this is 2D problem)
Q is not fixed on the conducting surfaces, resistivity is reciprocal of conductivity so the circles are not electrically isolated thus Q = Q(r)
Q is not a variable given in this problem.
This type of solution was considered "heavily-flawed" and was given 1 point. No exceptions will be made.

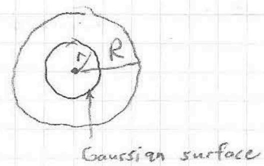
c) 1 point

Physics 7B Lec 1 Midterm 2, #5 Solution 4/2/2012

a) Find electric field using Gauss's Law

$$\int \vec{E}\cdot\hat{n}\,dA = \frac{Q_{enc}}{\varepsilon_0}$$

Inside: $r < R$



$$\int \vec{E}\cdot\hat{n}\,dA = E(4\pi r^2) \quad \text{via spherical symmetry}$$

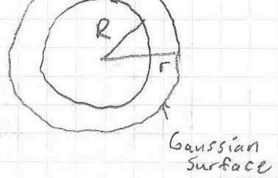
$$Q_{enc} = \tfrac{4}{3}\pi r^3 \rho \qquad \rho = \frac{Q}{\frac{4}{3}\pi R^3} = \text{charge density}$$

$$= Q\left(r^3/R^3\right)$$

$$\Rightarrow E(4\pi r^2) = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}$$

$$\boxed{E(r) = \frac{Qr}{4\pi\varepsilon_0 R^3}, \quad r < R}$$

Outside: $r > R$



$$\int \vec{E}\cdot\hat{n}\,dA = E(4\pi r^2)$$

$$Q_{enc} = Q$$

$$\Rightarrow E(4\pi r^2) = Q/\varepsilon_0$$

$$\boxed{E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad r > R}$$

Find energy density

$$u = \frac{\varepsilon_0}{2}E^2 = \frac{\varepsilon_0}{2}\begin{cases} \dfrac{Q^2 r^2}{(4\pi\varepsilon_0)^2 R^6} & r < R \\ \dfrac{Q^2}{(4\pi\varepsilon_0)^2 r^4} & r > R \end{cases}$$

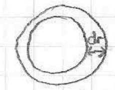
$$u = \begin{cases} \dfrac{Q^2 r^2}{32\pi^2\varepsilon_0 R^6} & r < R \\ \dfrac{Q^2}{32\pi^2\varepsilon_0 r^4} & r > R \end{cases}$$

Integrate energy density over volume to find total potential energy.

$$U = \int u\,dV$$

Use a spherical shell volume element

$$dV = 4\pi r^2 dr$$



Area = $4\pi r^2$

Volume ≈ $4\pi r^2 dr$

$$U = \int_0^\infty u\, 4\pi r^2 dr$$

Go from $r=0$ to $r=\infty$ to cover all space.

Break up into regions $r<R$ and $r>R$

$$U = \int_0^R u\, 4\pi r^2 dr + \int_R^\infty u\, 4\pi r^2 dr$$

$$= \int_0^R \frac{Q^2 r^2}{32\pi^2 \varepsilon_0 R^6} 4\pi r^2 dr + \int_R^\infty \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} 4\pi r^2 dr$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R^6} \int_0^R r^4 dr + \frac{Q^2}{8\pi\varepsilon_0} \int_R^\infty \frac{dr}{r^2}$$

$$= \frac{Q^3}{8\pi\varepsilon_0 R^6} \left.\frac{r^5}{5}\right|_0^R + \frac{Q^2}{8\pi\varepsilon_0} \left(-\frac{1}{r}\right)_R^\infty$$

$$= \frac{Q}{8\pi\varepsilon_0 R^6} \frac{R^5}{5} + \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{R}\right) = \frac{1}{5} \frac{Q^2}{8\pi\varepsilon_0 R} + \frac{Q^2}{8\pi\varepsilon_0 R}$$

$$= \left(\frac{1}{40} + \frac{1}{8}\right) \frac{Q^2}{\varepsilon_0 R} = \frac{1+5}{40} \frac{Q^2}{\varepsilon_0 R}$$

$$\boxed{U = \frac{3Q^2}{20\varepsilon_0 R} = \frac{3kQ^2}{5R}}$$

b) Find electric field

Inside: $r < R$

$$\boxed{E = 0 \quad \text{inside a conductor}}$$

$\int \vec{E} \cdot \hat{n}\, dA = 0 = Q_{enc}/\varepsilon_0 \quad \Rightarrow$ No charge inside the sphere. It is all on the surface.

Outside: $r > R$

$$\int \vec{E} \cdot \hat{n}\, dA = E(4\pi r^2)$$

$$Q_{enc} = Q$$

$$\Rightarrow \quad E(4\pi r^2) = Q/\varepsilon_0$$

$$\boxed{E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad r > R}$$

Find energy density

$$u = \frac{\varepsilon_0}{2} E^2 = \begin{cases} 0 & r < R \\ \dfrac{Q^2}{32\pi^2 \varepsilon_0 r^4} & r > R \end{cases}$$

Integrate over all space to find potential energy

$$U = \int u\, dV = \int_0^\infty u\, 4\pi r^2\, dr$$

$$= \int_0^R u\, 4\pi r^2\, dr + \int_R^\infty u\, 4\pi r^2\, dr$$

$$= 0 + \int_R^\infty \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} 4\pi r^2\, dr$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \int_R^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\right)_R^\infty = \frac{Q^2}{8\pi\varepsilon_0} \frac{1}{R}$$

$$\boxed{U = \frac{Q^2}{8\pi\varepsilon_0 R} = \frac{kQ^2}{2R}}$$

c) This problem helps illustrate why charge distributes itself uniformly over the surface of a conducting sphere.

Both the non-conducting sphere and conducting sphere had the same distribution of electric potential energy for $r > R$. However the non-conducting sphere had additional potential energy for $r < R$ whereas the conducting sphere had none. Thus, the non-conducting sphere had greater potential energy overall. Since charges are free to move in a conductor, we expect them to distribute in a way that minimizes the total potential energy. This can be done by making the potential energy density vanish inside the sphere for $r < R$, which requires $\vec{E} = 0$. By Gauss's law, this implies that all charge must lie on the surface.