Department of Mechanical Engineering
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ME185 - Introduction to Continuum Mechanics

Final Exam

- Please be sure to write your name on each sheet of paper that you use.
- Start each problem on a separate sheet.
- Clearly identify your answers and provide your supporting work so that we can follow your logic in arriving at the answer(s).

Problem 1 **(25 Points)**

Consider the motion of a deformable body $\mathcal{B}$ defined by

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + t\, f(X_3) \\ x_2 &= \chi_2(X_A, t) = X_2 + t\, f(X_3) \\ x_3 &= \chi_3(X_A, t) = X_3 \end{aligned} \qquad (1)$$

where $f(X_3)$ is a continuously differentiable function of its argument with derivatives f', f'', etc.

(i) Determine the components of the deformation gradient $\boldsymbol{F}$.

(ii) Is this motion isochoric?

(iii) Determine the velocity in spatial coordinates.

(iv) Is this motion steady?

(v) Determine the components of the right Cauchy-Green deformation tensor $\boldsymbol{C}$.

(vi) Find the unit vector $\boldsymbol{M}$ in the reference configuration such that any material line element aligned with $\boldsymbol{M}$ in the reference configuration experiences no stretch as a result of this deformation.

(vii) Determine the components of the velocity gradient tensor $\boldsymbol{L}$.

(viii) Calculate the material time derivative of the stretch λ of a line element which in the reference configuration lies in the direction $\frac{1}{\sqrt{2}}(\boldsymbol{E}_1 + \boldsymbol{E}_2)$.

Problem 2 (**25 Points**)

A new material is being tested, and a constitutive equation for the Cauchy stress tensor in terms of various kinematic measures has been proposed. The proposed relation is of the form

$$\boldsymbol{T} = a_1\boldsymbol{I} + a_2\boldsymbol{B} + a_3\boldsymbol{D} + a_4\boldsymbol{v}\otimes\boldsymbol{v} \tag{2}$$

where $\boldsymbol{T}$ is the Cauchy stress, $\boldsymbol{B}$ is the left Cauchy Green stretch tensor, $\boldsymbol{D}$ is the rate of stretch tensor, and $\boldsymbol{v}$ is the velocity vector. In this expression a_1, a_2, a_3 and a_4 are material constants.

(i) Determine the physical dimension for each of the material constants.

(ii) Do invariance requirements impose any restrictions on Equation (2)? If so, state them and write down the constitutive relation that results from their imposition.

Consider the simple shear motion given by the mapping

$$\boldsymbol{x} = (X_1 + \beta t X_2)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2 + X_3\boldsymbol{e}_3 \tag{3}$$

where β is a constant.

(iii) Determine the components of the Cauchy stress tensor with respect to the current basis $\{\boldsymbol{e}_i\}$.

(iv) Is simple shear possible in this material in the absence of body forces? Justify your answer.

(v) What restrictions must be imposed on the material constants a_1, a_2, a_3 and/or a_4 if the stress is to vanish when the body is in the reference configuration ($\beta = 0$)?

(vi) Using the restrictions determined above, can all of the coefficients a_1, a_2, a_3 and a_4 be determine from a single test in which the shear stress component T_{12} is measured as a function of time for a particular choice of $\beta \neq 0$? If so, show how this would be achieved. If not, what additional measurements would be necessary?

Problem 3 **(25 Points)**

Consider another material for which the constitutive equation is assumed to be of the form

$$T = \hat{T}(F, \dot{F}). \tag{4}$$

Use invariance arguments and the polar decomposition of the deformation gradient tensor $F = RU$ to deduce that

$$T = R\hat{T}(U, \dot{U})R^T, \tag{5}$$

where $\hat{T}$ is the same function in both equations.

Problem 4 **(25 Points)**

Let

$$j = \det(F^{-1}), \tag{6}$$

and recall that

$$\dot{j} = J\mathrm{div}\, v, \tag{7}$$

where

$$J = \det(F). \tag{8}$$

(i) Show that

$$\frac{\partial j}{\partial t} + \mathrm{div}(jv) = 0. \tag{9}$$

Note that the volume of a portion $\mathcal{P}_0$ of a body may be written as

$$\int_{\mathcal{P}_0} dV = \int_{\mathcal{P}} j dv. \tag{10}$$

(ii) Use Reynolds' transport theorem and the localization theorem to obtain Equation (9). Be sure to provide any restrictions and/or other information that are necessary in order to use these theorems.