EECS 20N: Midterm 2 Solutions

1

(a) The LTI system is not causal because its impulse response isn't zero for all time less than zero. See Figure 1.

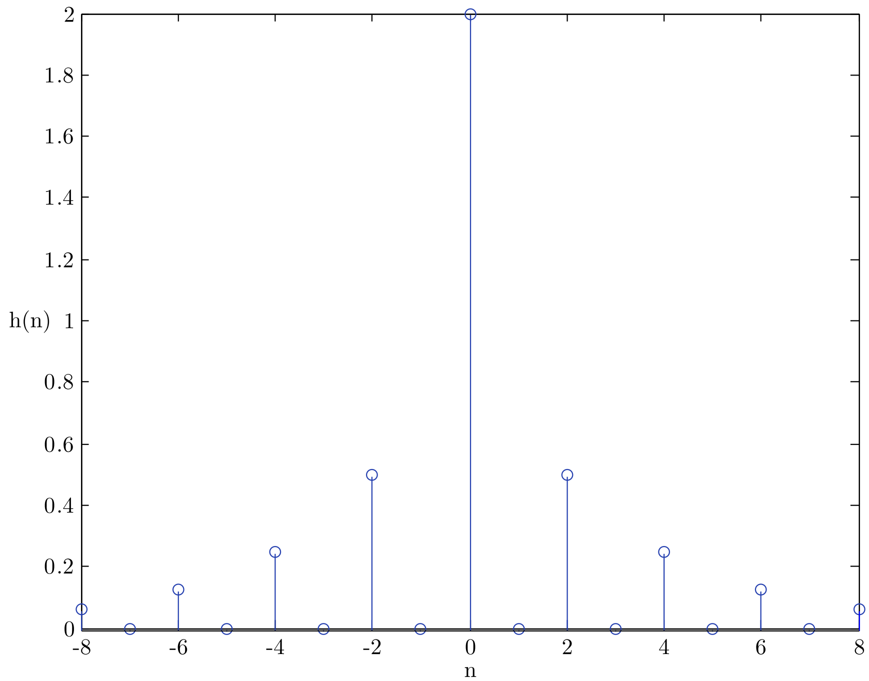

Figure 1: The system's impulse response in 1(a).

(b) Recall that the original impulse response is

$$h(n) = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ \alpha^{|n|/2} & \text{if } n \text{ is even,} \end{cases} \tag{1}$$

for some $\alpha \in (0, 1)$ (or in other words $0 < \alpha < 1$). We then evaluate the summation

$$\sum_{i=-\infty}^{\infty} h(n) e^{-i\omega n}. \tag{2}$$

We can then split the summation into three parts. Hence,

$$H(\omega) = \sum_{m=-\infty}^{-1} h(m) e^{-i\omega m} + 2 + \sum_{n=1}^{\infty} h(n) e^{-i\omega n} \tag{3}$$

$$= \sum_{n=1}^{\infty} h(-n) e^{i\omega n} + 2 + \sum_{n=1}^{\infty} h(n) e^{-i\omega n}, \tag{4}$$

where the second equality follows by setting $m = -n$. Note that a significant mistake made here was people assuming the FALSE fact that $e^{i\omega n} = e^{-i\omega n}$. Next, we note that for n odd, $h(n) = 0$, so that we can consider the summation only over even numbers. Thus,

$$\begin{aligned}
H(\omega) &= \sum_{n=1}^{\infty} h(-n)e^{i\omega n} + 2 + \sum_{n=1}^{\infty} h(n)e^{-i\omega n} && (5)\\
&= \sum_{n\geq 1 \text{ and } n \text{ even}} h(-n)e^{i\omega n} + 2 + \sum_{n\geq 1 \text{ and } n \text{ even}} h(n)e^{-i\omega n} && (6)\\
&= \sum_{k=1}^{\infty} h(-2k)e^{i\omega(2k)} + 2 + \sum_{k=1}^{\infty} h(2k)e^{-i\omega(2k)} && (7)\\
&= \sum_{k=1}^{\infty} \alpha^k e^{i\omega 2k} + 2 + \sum_{k=1}^{\infty} \alpha^k e^{-i\omega 2k} && (8)\\
&= \sum_{k=0}^{\infty} (\alpha e^{i\omega 2})^k + \sum_{k=1}^{\infty} (\alpha e^{-i\omega 2})^k && (9)\\
&= \frac{1}{1-\alpha e^{i2\omega}} + \frac{1}{1-\alpha e^{-i2\omega}} && (10)\\
&= \frac{2-2\alpha\cos(2\omega)}{1-2\alpha\cos(2\omega)+\alpha^2}. && (11)
\end{aligned}$$

Whence, the above chain of equalities yields the frequency response $H(\omega)$.

(c) For those of you who started from the cosine form for part 1(b), namely:

$$H(\omega) = \frac{2-2\alpha\cos(2\omega)}{1+\alpha^2-2\alpha\cos(2\omega)}.$$

You can simply plot the absolute value of the numerator and denominator separately and divide the first plot by the second to arrive at your final answer plotted in Figure 2, plugging in $\omega = 0, \pi, -\pi$ to arrive at the magnitudes at the important points.

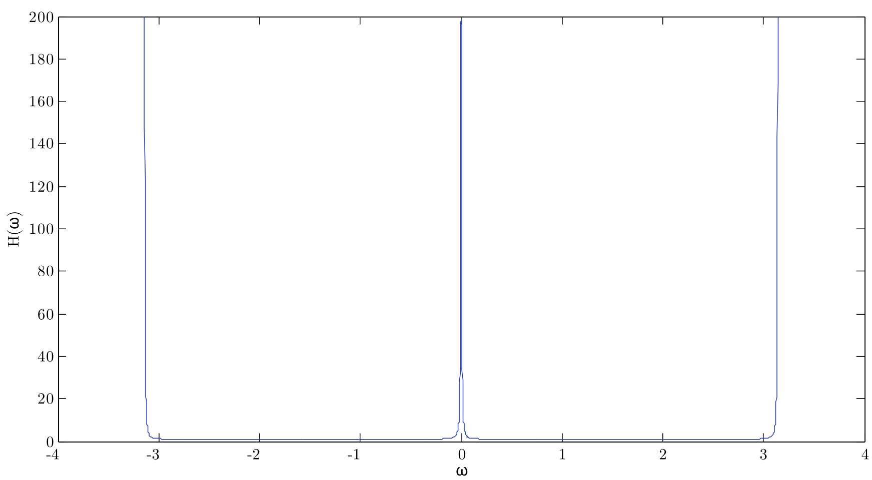

Figure 2: The system's frequency response for $\alpha = 0.99$ in 1(c). The important labels are $\omega = 0, \pi, -\pi$.

However, if you started from $H(\omega) = \frac{1}{1-\alpha e^{2i\omega}} + \frac{1}{1-\alpha e^{-2i\omega}}$, there's no easy way to arrive at the plot of Figure 2 without going to the cosine form.

Trying to find the Xs and Os of this function is extremely difficult by hand, and attempting to plot this function with the graphical method was thus pretty much a dead end (though it is theoretically doable with a considerable amount of algebra, but nobody successfully completed the problem this way).

The most common mistake was to assume that $|a+b| = |a| + |b|$. If you did this, you were only given half credit (5 points), even if it looks like the plot above. This is a very important mistake that you should watch for in the future. Specifically, many of you said that $H(\omega) = \left|\frac{1}{1-\alpha e^{2i\omega}}\right| + \left|\frac{1}{1-\alpha e^{-2i\omega}}\right|$. However, **this is NOT true**, though for this particular $H(\omega)$ it gives a cosmetically very similar plot.

Another somewhat common error was to set $\alpha = 1$. The problem statement says α is very close to 1 (for example 0.99). Though $\lim_{\alpha\to 1} H(\omega) = 2$, we did not want α to be exactly 1.

Note that if you had a different $H(\omega)$, as long as your plot is correct for your $H(\omega)$, you should receive full credit. And yes, I did write a MATLAB program to test our your $H(\omega)$ no matter how weird it was.

(d)

$$\begin{aligned}
\lim_{\alpha\to 0} |H(\omega)| &= \lim_{\alpha\to 0}\left|\frac{1}{1-\alpha e^{-i2\omega}} + \frac{1}{1-\alpha e^{i2\omega}}\right| \\
&= \left|\frac{1}{1-(0)e^{-i2\omega}} + \frac{1}{1-(0)e^{i2\omega}}\right| \\
&= |1+1| = 2
\end{aligned}$$

As $\alpha \to 0$, we see from the frequency response that the system appears to simply scale the input signal by 2.

From the impulse response, we can see that as $\alpha \to 0$, $h(n) \to 2\delta(n)$. We know that if the impulse response is $\delta(n)$, the output signal copies the input signal. By linearity, if we scale the impulse response by 2, the output signal also scales by 2.

Thus, the results we get are consistent.

2

(a)

$$\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt \\
&= \int_A^B Ce^{-i\omega t}dt \\
&= C\frac{1}{-i\omega}e^{-i\omega t}\Big|_A^B \\
&= \frac{C}{-i\omega}\left(e^{-i\omega B} - e^{-i\omega A}\right) \\
&= \frac{C}{i\omega}\left(e^{-i\omega A} - e^{-i\omega B}\right) \\
&= \frac{C}{i\omega}\left(e^{\frac{-i\omega A}{2}} - e^{-i\omega B}e^{\frac{i\omega A}{2}}\right)e^{\frac{-i\omega A}{2}} \\
&= \frac{C}{i\omega}\left(e^{\frac{-i\omega A}{2}}e^{\frac{i\omega B}{2}} - e^{\frac{-i\omega B}{2}}e^{\frac{i\omega A}{2}}\right)e^{\frac{-i\omega A}{2}}e^{\frac{-i\omega B}{2}} \\
&= \frac{C}{i\omega}\left(e^{\frac{-i\omega(A-B)}{2}} - e^{\frac{-i\omega(B-A)}{2}}\right)e^{\frac{-i\omega(B+A)}{2}} \\
&= \frac{2C}{\omega}\sin\left(\frac{B-A}{2}\omega\right)e^{\frac{-i\omega(B+A)}{2}}
\end{aligned}$$

(b) There are two ways do to this problem. The first would be to use the definition of the frequency response and directly evaluate

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = \int_{-T_2}^{-T_1} e^{-i\omega t}dt + \int_{-T_1}^{T_1} 2e^{-i\omega t}dt + \int_{T_2}^{T_1} e^{-i\omega t}dt.$$

This method is a bit tedious but, of course, produces the correct answer. However, this is a much faster way to do this problem. First, define the impulse response

$$h_1(t) = \begin{cases} 1 & \text{if} \quad |t| \leq T_1 \\ 0 & \text{if} \quad |t| > T_1 \end{cases}$$

and similarly

$$h_2(t) = \begin{cases} 1 & \text{if} \quad |t| \leq T_2 \\ 0 & \text{if} \quad |t| > T_2 \end{cases} .$$

Note that $h(t) = h_1(t) + h_2(t)$. Since h_1 is a special case of the function in the previous part with $A = -T_1$ and $B = T_1$, we have that

$$\begin{aligned} H_1(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \\ &= \tfrac{2C}{\omega}\sin(\tfrac{T_1+T_1}{2}\omega)e^{\frac{-i\omega(T_1-T_1)}{2}} \\ &= \tfrac{2C}{\omega}\sin(T_1\omega). \end{aligned}$$

Similarly, $H_2(\omega) = \frac{2C}{\omega}\sin(T_2\omega)$.

Finally, we calculate:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} h_1(t)e^{-i\omega t}dt + \int_{-\infty}^{\infty} h_2(t)e^{-i\omega t}dt \\ &= H_1(\omega) + H_2(\omega) \\ &= \tfrac{2C}{\omega}\sin(T_1\omega) + \tfrac{2C}{\omega}\sin(T_2\omega) \\ &= \tfrac{2C}{\omega}(\sin(T_1\omega) + \sin(T_2\omega)) \\ &= \tfrac{4C}{\omega}\sin(\tfrac{T_1+T_2}{2}\omega)\cos(\tfrac{T_2-T_1}{2}\omega). \end{aligned}$$

Thus, we have:

$$\begin{aligned} \alpha &= 2C \\ \xi &= 4C \\ \beta &= T_1 \\ \gamma &= T_2 \\ \lambda &= T_1 + T_2 \\ \mu &= T_2 - T_1. \end{aligned}$$

(c)(i) Using $T_1 = 49s$ and $T_2 = 51s$:

$$H(\omega) = \frac{4C}{\omega}\sin(50\omega)\cos(\omega).$$

Notice that this is a high-frequency sinc modulated by a cosine. Its zero crossings are when:

$$\begin{aligned} \sin(50\omega) = 0 \quad &\rightarrow \quad \omega = \frac{k\pi}{50} && \text{for } k \in \mathbb{Z}, \text{and} \\ \cos(\omega) = 0 \quad &\rightarrow \quad \omega = \frac{\pi}{2} + k\pi && \text{for } k \in \mathbb{Z}. \end{aligned}$$

We can calculate $H(0)$ using L'Hôpital's rule:

$$\begin{aligned} H(0) &= \lim_{\omega\to 0}\frac{4C}{\omega}\sin(50\omega)\cos(\omega) \\ &= \lim_{\omega\to 0} 4C\left[\cos(50\omega)50\cos(\omega) + \sin(50\omega)(-\sin(\omega))\right] \\ &= 200C \end{aligned}$$

Finally, assuming $C = 1$, $H(\omega)$ is plotted in Figures 3 and 4.

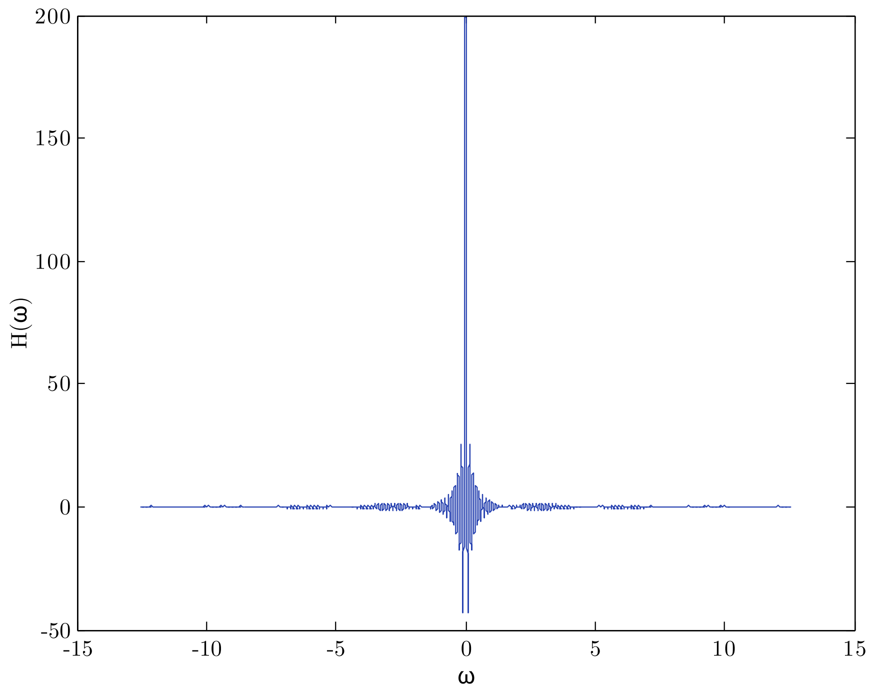

Figure 3: Plot of $H(\omega)$ for $|\omega| \leq 4\pi$ in 2(c)(i). Note how the sinc peaks at $H(0) = 200$.

(c)(ii) LTI system H has the following input-output pair:

$$e^{i\omega t} \xrightarrow{H} H(\omega)e^{i\omega t} \tag{12}$$

We can express input $x(t)$ as a sum of complex exponentials:

$$\begin{aligned} x(t) =& \sum_{k=0}^{\infty} D_k \cos\left(k\frac{\pi}{50}t\right) + \sum_{l=1}^{\infty} E_l \sin\left(l\frac{\pi}{2}t\right) \\ =& \sum_{k=0}^{\infty} \frac{D_k}{2}\left[e^{ik\frac{\pi}{50}t} + e^{-ik\frac{\pi}{50}t}\right] + \sum_{l=1}^{\infty} \frac{E_l}{2i}\left[e^{il\frac{\pi}{2}t} - e^{-il\frac{\pi}{2}t}\right] \end{aligned}$$

To be explicit, the input-output pair for frequency component $e^{i\frac{k\pi}{50}t}$ would be:

$$e^{ik\frac{\pi}{50}t} \xrightarrow{H} H\left(k\frac{\pi}{50}\right)e^{ik\frac{\pi}{50}t}$$

And similarly for the other frequency components. By superposition, the output for input $x(t)$ is:

$$y(t) = \sum_{k=0}^{\infty} \frac{D_k}{2}\left[H\left(k\frac{\pi}{50}\right)e^{ik\frac{\pi}{50}t} + H\left(-k\frac{\pi}{50}\right)e^{-ik\frac{\pi}{50}t}\right] + \sum_{l=1}^{\infty} \frac{E_l}{2i}\left[H\left(l\frac{\pi}{2}\right)e^{il\frac{\pi}{2}t} - H\left(-l\frac{\pi}{2}\right)e^{-il\frac{\pi}{2}t}\right]$$

The above does not rely on any of the previous question parts. From part 2(c)(i), we have that $H(\omega)$ has zero-crossings at $k\frac{\pi}{50}, k \in \mathbb{Z}$, except $k = 0$ where $H(0) = 200$. This means almost all of the above $H(\cdot)$ terms are zero, including $H\left(\pm l\frac{\pi}{2}\right)$ since $l\frac{\pi}{2} = k\frac{\pi}{50}$ for $k = 25l$. Thus the only terms remaining are:

$$\begin{aligned} y(t) =& \frac{D_0}{2}\left[H(0)e^{i0t} + H(0)e^{i0t}\right] \\ =& D_0 H(0) = 200D_0 \end{aligned}$$

Feedback:

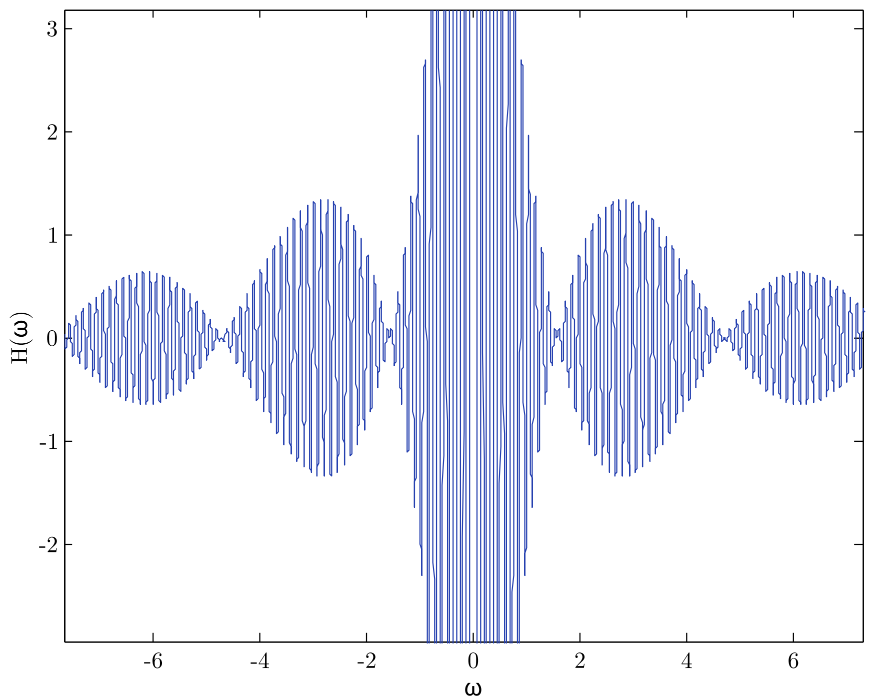

Figure 4: Detail of Figure 3 from 2(c)(i). Note the zero crossings every $\frac{k\pi}{50}$ for $k \in \mathbb{Z}$ and the cosine modulation for $|\omega| \leq 2\pi$.

- Many students continue to write $y(t) = H(\omega)x(t)$ in some form or another, such as:

$$y(t) = H(\omega)\left[\sum_{k=0}^{\infty} D_k \cos\left(k\frac{\pi}{50}t\right) + \sum_{l=1}^{\infty} E_l \sin\left(l\frac{\pi}{2}t\right)\right] \tag{13}$$

  This is incorrect because 1) equation (12) applies when the input is a single pure complex exponential, but the input here is a mix of complex exponentials; and 2) it is unknown what ω represents.

- The convolution method of finding the output is only practical if the integral (or summation in discrete-time) is feasible in the time and space allowed. Otherwise, you are better off using a different approach.

- Given $H(\omega) = \frac{4}{\omega}\sin(50\omega)\cos(\omega)$ from part 2(c)(i), many students said that the $\sin(50\omega)$ term would kill off all $e^{ik\frac{\pi}{50}t}$ terms (including $k = 0$, which is not true) while only looking at the $\cos(\omega)$ for the $e^{il\frac{\pi}{2}t}$ terms, which lead them to say even l terms were passed. Moral: look at the whole frequency response expression.

- The following is, in general, not true:

$$\cos(\omega t) \xrightarrow{H} H(\omega)\cos(\omega t)$$

  because cosine and sine contain multiple frequency components. The following is true:

$$\cos(\omega t) = \frac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right] \xrightarrow{H} \frac{1}{2}\left[H(\omega)e^{i\omega t} + H(-\omega)e^{-i\omega t}\right]$$

- Dummy variables should not remain after the summation (or integral) is evaluated, so for this problem there should be no k or l remaining.

- “Attenuate” means to reduce, not eliminate. You keep using that word. I do not think it means what you think it means.