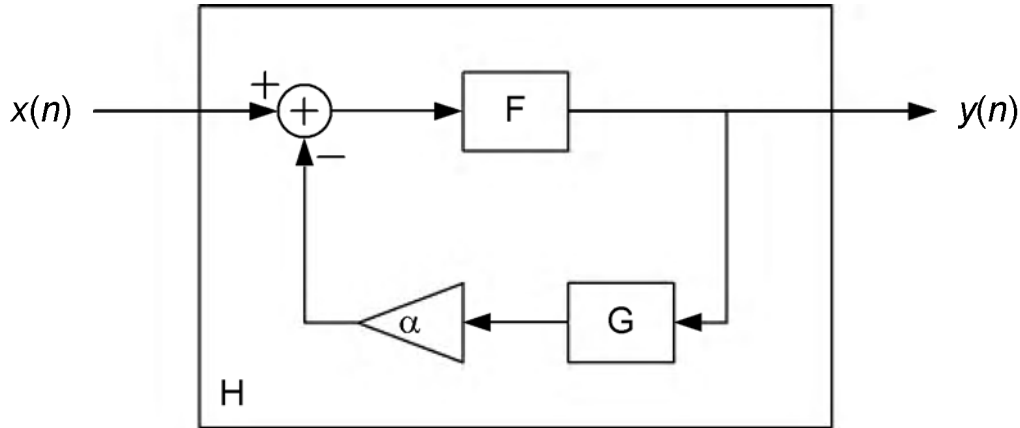


LAST Name Angell FIRST Name Tri  
Lab Time ?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—including, for example, commencing work prematurely or continuing beyond the announced stop time—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

**MT3.1 (30 Points)** Consider the discrete-time feedback system  $H$ , as shown in the figure below. The LTI system  $F$  is called the *plant*. The LTI controller consists of the cascade of an LTI system  $G$  and a tunable multiplier whose scaling factor is  $\alpha$ .



The plant is BIBO stable and its impulse response is

$$f(n) = \begin{cases} \beta^{n/2} & \text{if } n \text{ is even and nonnegative,} \\ 0 & \text{if } n \text{ is odd or negative.} \end{cases}$$

Furthermore, the filter  $G$  has impulse response  $g(n) = \delta(n - 2)$ , for all  $n$ .

For each part below, you must explain your reasoning succinctly, but clearly and convincingly.

(a) (10 Points) Show that the frequency response of the plant is given by

$$F(\omega) = \frac{1}{1 - \beta e^{-i2\omega}}$$

The impulse response of the plant can be thought of as the upsampled version of the function  $q$  described by  $q(n) = \beta^n u(n)$ . In particular,  $f(n) \xrightarrow{\uparrow 2} f(n)$ . As such,  $F(\omega) = Q(2\omega)$ ,

where  $Q(\omega) = \frac{1}{1 - \beta e^{-i\omega}}$ . We can show this as follows:

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega n} = \sum_{\substack{n=0 \\ n=\text{even}}}^{\infty} \beta^{n/2} e^{-i\omega n} = \sum_{l=0}^{\infty} \beta^{2l/2} e^{-i\omega 2l} = \sum_{l=0}^{\infty} (\beta e^{-i2\omega})^l$$

let  $n=2l$

We're told that the plant  $F$  is BIBO stable  $\Rightarrow |\beta| < 1 \Rightarrow |\beta e^{-i2\omega}| < 1$ , which enables us to determine a closed-form expression for  $\sum_{l=0}^{\infty} (\beta e^{-i2\omega})^l$ :  $F(\omega) = \frac{1}{1 - \beta e^{-i2\omega}}$

(b) (5 Points) Determine the frequency response  $G(\omega)$  of the filter  $G$ .

$$g(n) = \delta(n-2) \Rightarrow G(\omega) = \sum_{n=-\infty}^{\infty} \delta(n-2) e^{-i\omega n} \Rightarrow G(\omega) = e^{-i2\omega}$$

We can also obtain this by knowing  $\delta(n) \xrightarrow{\mathcal{F}} 1$  and using the time-shift property of the DTFT.

(c) (10 Points) Determine the frequency response  $H(\omega)$  of the feedback system  $H$ .

$$H(\omega) = \frac{\text{Forward Gain}}{1 - \text{Loop Gain}} = \frac{F(\omega)}{1 + \alpha F(\omega) G(\omega)} = \frac{\frac{1}{1 - \beta e^{-i2\omega}}}{1 + \frac{\alpha e^{-i2\omega}}{1 - \beta e^{-i2\omega}}} \Rightarrow$$

$$H(\omega) = \frac{1}{1 - (\beta - \alpha) e^{-i2\omega}}$$

(d) (5 Points) For what values of the tunable parameter  $\alpha$  is the closed-loop system  $H$  BIBO stable?

The impulse response of the system  $H$  would have to be 
$$h(n) = \begin{cases} (\beta - \alpha)^{|n|/2} & n \geq 0 \text{ \& n=even} \\ 0 & n < 0 \text{ or } n=\text{odd} \end{cases}$$

The system  $H$  is BIBO stable if, and only if,  $|\beta - \alpha| < 1$ . In the complex plane, this simply means that  $\alpha$  must be strictly within a circle of radius 1 centered at  $\beta$ .

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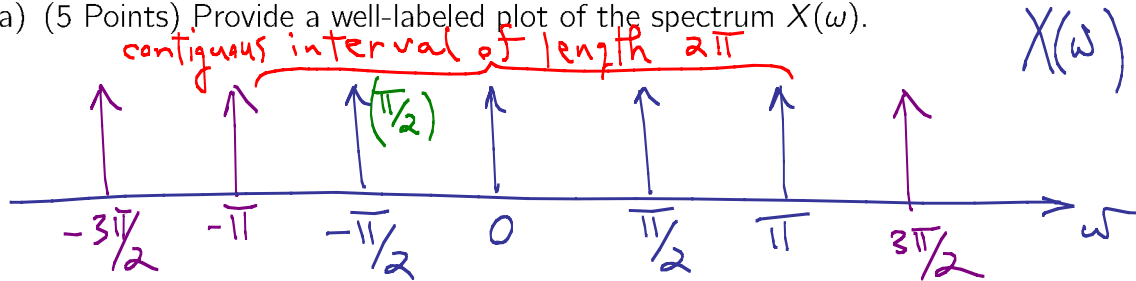


**MT3.2 (25 Points)** The DTFT of a signal  $x$  is given by

$$X(\omega) = \frac{\pi}{2} \sum_{\ell=-\infty}^{+\infty} \delta\left(\omega - \frac{\ell\pi}{2}\right).$$

For every part below, be sure to explain your reasoning succinctly, but clearly and convincingly.

(a) (5 Points) Provide a well-labeled plot of the spectrum  $X(\omega)$ .



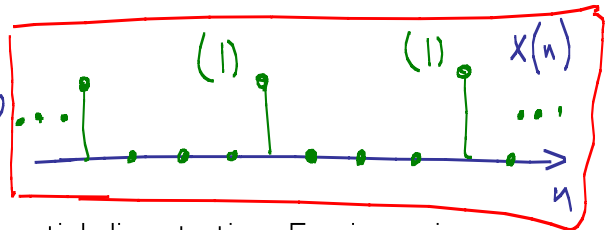
(b) (15 Points) Determine a reasonably simple expression for, and provide a well-labeled plot of, the signal  $x$  in the time domain. The signal  $x$  that you obtain must be periodic.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{i\omega n} d\omega = \frac{1}{2\pi} \frac{\pi}{2} \int_{-\pi}^{\pi} [\delta(\omega + \pi/2) + \delta(\omega) + \delta(\omega - \pi/2) + \delta(\omega - \pi)] e^{i\omega n} d\omega$$

means the impulse  $e^{-i\pi n}$  is not included but the one at  $\pi$  is.

$$x(n) = \frac{1}{4} (e^{-i\pi/2 n} + 1 + e^{i\pi/2 n} + e^{-i\pi n})$$

$$x(n) = \frac{1+(-1)^n}{4} + \frac{1}{2} \cos\left(\frac{\pi n}{2}\right) = \begin{cases} 1 & \text{if } n \bmod 4 = 0 \\ 0 & \text{otherwise.} \end{cases}$$



(c) (5 Points) Determine the complex exponential discrete-time Fourier series coefficients  $X_k$  for the signal  $x$ . That is, determine the coefficients  $X_k$  in  $x(n) = \sum_{k=\langle p \rangle} X_k e^{ik\omega_0 n}$ , where  $p$  is the fundamental period of the signal. Depending on how you tackled part(b), you may have to do a minuscule amount of work for this part.

Method 1:

$$x(n+4) = x(n) \quad \forall n \Rightarrow x(n) = \sum_{k=0}^3 X_k e^{ik\frac{2\pi}{4}n} = \sum_{k=0}^3 X_k e^{ik\frac{\pi}{2}n}$$

$$X_k = \frac{1}{4} \sum_{n=0}^3 x(n) e^{-ik\frac{\pi}{2}n} = \frac{1}{4} x(0) = \frac{1}{4} \Rightarrow X_k = \frac{1}{4} \quad \forall k \in \mathbb{Z}$$

Method 2:

$$e^{i\omega_0 n} \xleftrightarrow{F} 2\pi \delta(\omega - \omega_0) \quad \omega, \omega_0 \in (-\pi, \pi]$$

$$x(n) = \sum_{k=-1}^2 X_k e^{ik\omega_0 n} \Rightarrow X(\omega) = \sum_{k=-1}^2 2\pi X_k \delta(\omega - k\omega_0)$$

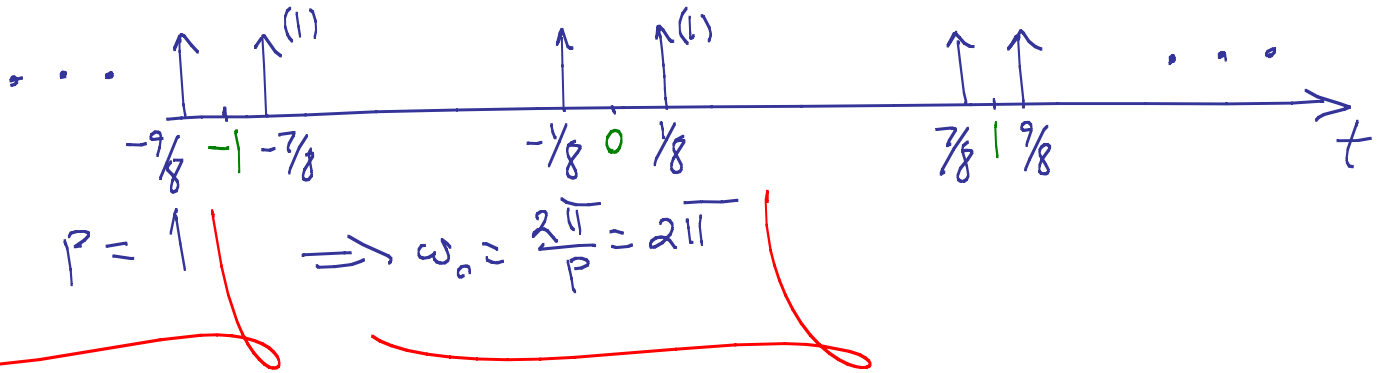
$$2\pi X_k = \frac{\pi}{2} \Rightarrow X_k = \frac{1}{4} \quad k = -1, 0, 1, 2$$

$\omega \in (-\pi, \pi]$   
 $2\pi$ -Periodically repeats outside

**MT3.3 (20 Points)** A periodic continuous-time signal  $x$  is described as follows:

$$x(t) = \sum_{\ell=-\infty}^{+\infty} \left[ \delta\left(t + \frac{1}{8} - \ell\right) + \delta\left(t - \frac{1}{8} - \ell\right) \right].$$

(a) (6 Points) Provide a well-labeled plot of the signal  $x$ , and determine its fundamental period  $p$  and fundamental frequency  $\omega_0$ .



(b) (14 Points) Determine the complex exponential Fourier series coefficients of the signal  $x$ . That is, determine  $X_k$  in the representation

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{ik\omega_0 t}.$$

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt = \frac{1}{1} \int_{-1/2}^{1/2} \left[ \delta\left(t + \frac{1}{8}\right) + \delta\left(t - \frac{1}{8}\right) \right] e^{-ik2\pi t} dt$$

$$\Rightarrow X_k = e^{ik2\pi/8} + e^{-ik2\pi/8} = 2 \cos\left(\frac{\pi}{4}\right) = 2 \left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \Rightarrow$$

$$X_k = \sqrt{2} \quad x(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} e^{ik2\pi t}$$

**MT3.4 (30 Points)** A discrete-time LTI filter  $G$ , shown below, has input  $x$ , corresponding output  $y$ , impulse response  $g$ , and frequency response  $G$ . This filter's impulse response is nonnegative; that is,  $g(n) \geq 0$ , for all  $n$ .



(a) (10 Points) We apply the constant input signal  $x(n) = 1$  to the filter. Determine the corresponding output signal  $y$ , and show that it's nonnegative ( $y(n) \geq 0$ , for all  $n$ ). Your expression for  $y(n)$  must be in terms of the filter's frequency response.

$$x(n) = 1 = e^{i0n} \Rightarrow y(n) = G(0) e^{i0n} = G(0). \quad \text{But } G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-i\omega n}$$

$$\Rightarrow G(0) = \sum_{n=-\infty}^{\infty} g(n). \quad \text{So, } y(n) = G(0) = \sum_{n=-\infty}^{\infty} g(n)$$

But  $g(n) \geq 0 \quad \forall n \Rightarrow G(0) = \sum_{n=-\infty}^{\infty} g(n) \geq 0 \Rightarrow y(n) \geq 0$

(b) (10 Points) Show that, in the frequency interval  $[-\pi, +\pi)$ , the magnitude response of the filter achieves a global maximum at  $\omega = 0$ . In other words, show that

$$|G(0)| \geq |G(\omega)|, \quad \forall \omega \in [-\pi, +\pi).$$

See the hint at the bottom of the next page.

$$G(0) = \sum_{n=-\infty}^{\infty} g(n) \Rightarrow |G(0)| = \left| \sum_n g(n) \right| = \sum_n |g(n)| = \sum_n |g(n) e^{-i\omega n}|$$

b/c  $g(n) \geq 0 \quad \forall n$

$$\geq \left| \sum_n g(n) e^{-i\omega n} \right| = |G(\omega)|$$

triangle inequality  $\uparrow$

$$\Rightarrow |G(0)| \geq |G(\omega)|$$

We know  $G(0) \geq 0 \Rightarrow$

$$G(0) \geq |G(\omega)|$$

- (c) (10 Points) For this part only, assume that the impulse response of the filter is *strictly positive* ( $g(n) > 0$ , for all  $n$ ). Show that, in the frequency interval  $[-\pi, +\pi)$ , the frequency response has a *unique* global maximum at  $\omega = 0$ . In other words, show that

$$|G(0)| > |G(\omega)|, \quad \forall \omega \in [-\pi, +\pi) - \{0\}.$$

In the previous part, the global maximum may not have been unique, but it is here, given the stricter condition that we've imposed on the impulse response  $g$ .

If  $g(n) > 0$ , then for  $\omega$  such that  $\omega \bmod{2\pi} \neq 0$ ,  $g(n)e^{-i\omega n}$  and  $g(m)e^{-i\omega m}$  cannot have the same phase if  $m \neq n$ .  
Hence, the inequality shown in part (b) is strict, according to the statement of the triangle inequality.

Hint: In parts (b) and (c), you may find it helpful to use the following version of the triangle inequality for complex numbers (don't try to prove this result):

Given a (possibly infinite) set of complex numbers  $z_k = R_k e^{i\theta_k}$ , it's true that

$$\left| \sum_k z_k \right| \leq \sum_k |z_k|,$$

where equality holds if, and only if, the numbers  $z_k$  have the same phase (i.e.,  $\theta_k = \theta$ , for all  $k$ ).

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Problem Name	Points	Your Score
	10	10
1	30	30
2	25	25
3	20	20
4	30	30
<b>Total</b>	<b>115</b>	<b>115</b>

Potentially useful formulas:

DTFT:

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{i\omega n} d\omega \quad \longleftrightarrow \quad X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-i\omega n}$$

DTFS:

$$x(n) = \sum_{k=\langle p \rangle} X_k e^{ik\omega_0 n} \quad \longleftrightarrow \quad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) e^{-ik\omega_0 n}$$

CTFS:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{ik\omega_0 t} \quad \longleftrightarrow \quad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt$$