Operations Research II, IEOR161

University of California, Berkeley

Spring 2007 Final Exam

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7 questions.

1. **[5+5]** Let X and Y be independent exponential random variables where X has rate μ and Y has rate λ. Calculate $\mathbb{E}[min(X, Y)]$ and $\text{Var}(min(X, Y))$.

2. **[10+10]** Jack the bomb disposal expert finds two live bombs in an abandoned warehouse in the outskirts of Los Angeles. The first bomb will explode at an exponential time with rate μ_1 and the second at an exponential time with rate μ_2. Jack takes an exponential amount of time with rate λ to disarm a bomb (regardless of the bomb). The lifetimes of both bombs and the time taken to disarm each of them are independent.

   (a) Suppose that Jack begins by trying to disarm bomb 1 and moves on to bomb 2 if he is successful with bomb 1 (and bomb 2 has not exploded). What is the probability that both bombs are disarmed before either of them explodes? If, in addition, it is known that $\mu_1 > \mu_2$ does it make any difference which bomb Jack tries disarming first? If so, which order should he disarm the bombs?

   (b) Suppose now that there are two experts, Jack and Doyle, who are capable of disarming bombs, and that Jack can disarm either bomb in an exponential time with rate λ_1, while Doyle takes an exponential time with rate λ_2. If $\lambda_1 > \lambda_2$ (Jack is more skilled), $\mu_1 > \mu_2$ (the first bomb is more likely to explode before the second), and Jack and Doyle must work concurrently, which bomb should Jack work on? (Does it make any difference?)[1]

3. **[10+10]** Patrons arrive at a store according to a Poisson process with rate λ. Assume that there are no customers when the store opens at 9am, that each patron spends an exponential amount of time at the store with rate θ, that the store has infinite capacity, and that the shopping times for each customer and the arrival processes are all independent.

   (a) Suppose that exactly 2 customers arrived between 9am and 10am. What is the probability that they are both still in the store at 10am?

   (b) What is the steady state distribution for the number of customers in the system?

[1] Assume that circumstances are such that both bombs are next to each other, that it is too dangerous to move either bomb, and that the consequences of either bomb exploding are enormous (e.g. they are nuclear devices). So answers like "The best thing to do is to take both bombs and throw them in the river" or "Leave Doyle to disarm both bombs sequentially since this will keep Jack alive for the next episode" are not valid answers. Assume that if Jack finishes before Doyle (say) he is not allowed to help Doyle disarm his bomb, and vice versa.

4. **[10]** Due to the stress of coping with business, Harry begins to experience migraine headaches of random severities. The times when headaches occur follow a Poisson process of rate λ. Headache severities $\{H_i\}$ are independent of times of occurrences and are iid exponential random variables with mean 1. (i.e. H_1 is the severity of headache 1, H_2 the severity of headache 2, etc). Assume that headaches are instantaneous and have duration 0. Harry decides that he will commit himself to hospital if a headache of severity greater than $c > 0$ occurs in the period of time $[0, t]$. Compute the probability that Harry commits himself during this time.

5. **[10+5]** A boy and girl move into a two-bar town on the same day. Each night the boy visits one or the other of the two bars, starting in bar 1, according to a Markov chain with transition matrix

$$\begin{bmatrix} .7 & .3 \\ .3 & .7 \end{bmatrix}$$

Likewise, the girl, visits one or the other of the two bars according to a Markov chain with transition matrix

$$\begin{bmatrix} .4 & .6 \\ .6 & .4 \end{bmatrix}$$

but starting in bar 2. Assume that the two Markov chains are independent. Naturally, the game ends when boy meets girl, i.e. when they go to the same bar.

   (a) Argue that the progress of the game can be described by a three state Markov chain where one state (representing the end of the game) is absorbing. Identify the states and the transition probability matrix.

   (b) Let N denote the number of the night on which boy meets girl. What is the distribution of N?

6. **[10]** Customers arrive at a queueing system according to a Poisson process with rate λ. There are k servers. Service times for each server are exponential and independent with rate μ. An arriving customer who finds a free server goes directly into service. Customers who find all servers busy depart from the system and never return (i.e. there is no queue of customers waiting for service). What is the probability that an arriving customer finds the system full (and hence leaves immediately)?

7. **[5+5+5+5+5]** Customers arrive at a 2 server system according to a Poisson process (rate λ). An arrival finding the system empty is likely to enter service with either server. An arrival finding one customer in the system will enter service with the idle server. An arrival finding 2 customers in the system will wait in line for the first free server. An arrival finding 3 in the system will not enter. All service times are exponential with rate μ, and once a customer us served (by either server), he departs the system.

   (a) Define the states

   (b) Find the long-run (steady state) probabilities

   (c) Suppose a customer arrives and finds 2 others in the system. What is the expected time he spends in the system?

   (d) What proportion of customers enter the system?

   (e) What is the average time an entering customer spends in the system?