Operations Research II IEOR161

University of California, Berkeley

Spring 2008

Final Exam

Name:

Student ID:

6 questions, total score is 150.

1. **[5, 5, 5]** A machine's revenue generating potential depends on its condition. Assume that there are 5 possible operating modes, {broken, poor, good, very good and excellent}, and that a new machine is in an "excellent" condition to start with, and deteriorates over time (to "very good", and then to "good", and then to "poor", before it breaks). Assume that the time between changes in condition are exponential with rate $\lambda = 0.5$ per hour (i.e. mean of 2 hours), and that the machine continues generating revenue until it is "broken". The machine generates revenue at the rate of \$10 per hour when its condition is "very good" or "excellent", and at \$5 per hour when its condition is "good". It drops to \$2 per hour when its condition is "poor".

   (a) What is the expected profit of the machine during its lifetime?

   (b) What is the probability that the machine remains in an "excellent" working condition for the first hour.

   (c) What is the probability that the machine is still in an "excellent" working condition if it is inspected at a time corresponding to an exponential random variable with mean $1/\mu = 10$ hours?

2. **[5, 5]** Customers arrive at a store according to a Poisson process with rate $\lambda = 2$ per hour.

   (a) What is the probability that there are no arrivals in the first and last hour of a given 10 hour interval?

   (b) Given that 5 customers arrived during this 10 hour interval, what is the probability that *exactly* one customer arrived in the first hour, and *exactly* one arrived in the last hour?

3. **[20]** Write down the balance equations and calculate the stationary distribution for a discrete time Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

4. **[5, 10, 10, 5]** A property agent has 100 identical homes in his portfolio and needs to choose a price y at which to sell each of them. Customers arrive at his office according to a Poisson process at a rate of λ customers per day. Customer j will make a purchase at price y if his/her *reservation price* X_j exceeds the asking price (i.e. $X_j > y$), and leaves without making a purchase if $X_j \leq y$. Assume that the reservation prices $X_1, X_2, \cdots$ are exponential random variables with mean μ (i.e. rate $1/\mu$) independent of each other as well as the arrival process, and that there are infinitely many potential customers (i.e. they continue arriving at rate λ until all the homes are sold).

   (a) What is the probability that any given customer makes a purchase when the selling price is set to y?

   (b) What is the distribution of the inter-arrival time between customers that make purchases? Hint: There are two types of customers, those that make a purchase and those that don't.

   (c) Assuming an operating cost of c dollars per day for the agent until all the homes have been sold, derive an expression for the expected profit from selling all the homes as a function of the selling price y.

   (d) Compute the optimal selling price y^* as well as the associated optimal profit for the property agent.

5. **[10+10, 10, 5+5+5+5]** Customers arrive at an exhibit according to a Poisson process with rate λ. There are n marketing agents at the exhibit. Each agent can talk to one customer at a time, which takes an exponential time with rate μ. (Customers immediately leave the exhibit after their service is complete). Any customer finding all marketing agents busy immediately departs without talking to anyone. Assume that customer arrivals process and the service times spent by each agent are independent.

   (a) If an arrival finds all servers busy, find

      i. The expected number of available agents found by the next arriving customer

      ii. The probability that the next arriving customer finds all the marketing agents free.

   (b) Model this system as a continuous time Markov chain; identify the states, and work out the transition rates between states.

   (c) Consider the special case when there are two marketing agent working at the exhibit such that customers arrive at rate $\lambda = 5$ per hour, and agents work at rate $\mu = 10$ per hour.

      i. Write down the balance equations?

      ii. What is the stationary distribution?

      iii. What proportion of potential customers are lost?

      iv. What is the average number of customers in the system?

6. **[5+10+5+5]** A small gas station has space for 2 cars. Potential customers arrive according to a Poisson process with rate $\lambda = 0.05$ per minute (i.e. with a mean interarrival time of 20 minutes), but will not stop if both spaces are filled (and are lost). There is one attendant whose service time is exponential with mean 5 minutes. The attendant serves one customer at a time (in the order that they arrived).

   (a) If a given potential customer finds the gas station full, what is the probability that the next 2 potential customers will also find it full?

   (b) Calculate the stationary distribution and use this to determine the fraction of potential customers that are lost.

   (c) What is the expected number of customers in the gas station?

   (d) What is the expected waiting time (in the system) of those who are eventually served?